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On the quasi-extended addition for exploded real numbers

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Abstract. In teaching primary teacher trainees, an awareness of the characteristic features, especially commutativity and associativity of basic operations play an important role. Owing to a deeply set automatism rooted in their primary and secondary education, teacher trainees think that such characteristics of addition are so trivial that they do not need to be proved. It does not cause a difficulty in applying mathematical knowledge in everyday situations but primary teachers must have a deeper insight. That is why it is reasonable to show these characteristic features to primary teacher trainees in a different algebraic structure. An example for that could be the algebra of vectors. In this paper the algebraic structure of exploded numbers containing the set of real numbers as a subset is selected as an example. With the help of super-operations (super-addition, super-multiplication, super-subtraction and super-division) introduced for exploded numbers, we try to extend addition for exploded numbers as well. The question of the method of extension and the examination of the characteristics of the extended addition arises. While seeking for the answer, surprising facts emerge, such as the phenomenon that each real number will have one and only one addition incompetent pair among exploded numbers. In this paper we introduce the quasi-extended addition for exploded real numbers which is essentially different from super - addition. On the other hand, the quasi-extended addition is the (traditional) addition for real numbers. Moreover, we investigate some properties (for example commutativity, associativity) of quasi-extended addition. Finally, we find some similarity between the countable infinity and the exploded of 1. The quasi-extension of addition is useful for students to observe different kinds of algebraic properties, too.

Zusammenfassung. Bei der Ausbildung von Grundschullehrtrainees spielen ein Bewusstsein der charakteristischen Eigenschaften, besonders Kommutativitt und Assoziativitt von Grundrechenarten eine wichtige Rolle. Dank einem tiefsitzenden Automatismus, der in ihrer Grundausbildung und weiteren Ausbildung verwurzelt ist, denken Lehrertrainees, dass solche Charakteristika der Addition so trivial sind, dass sie nicht bewiesen werden müssen. Es verursacht keine Schwierigkeit in der tglichen Anwendung der mathematischen Kenntnis in tglichen Situationen, aber die Grundschullehrer benötigen eine tiefere Sicht. Deshalb ist es sinnvoll, diese charakteristischen Eigenschaften den angehenden Grundschullehrern in einer anderen algebraischen Struktur zu zeigen. Ein Beispiel dafür könnte die Vektoralgebra sein. In diesem Papier wird die algebraische Struktur der ansteigenden Zahlen, die den Satz der realen Zahlen enthlt, als Beispiel ausgewhlt. Mithilfe der Super-Operationen (Super-Addition, Super-Multiplikation , Super-Subtraktion und

Super-Division), die für ansteigende Zahlen eingeführt wurde, versuchen wir, die Addition für ansteigende Zahlen ebenso auszudehnen. Es stellt sich die Frage nach der Ausdehnungsmethode und der Prüfung der Charakteristika. Whrend man die Antwort sucht, tauchen überraschende Fakten auf, wie das Phnomen, dass jede reale Zahl ein, nur ein einziges additionsuntaugliches Paar unter den ansteigenden Zahlen haben wird. In diesem Papier führen wir die quasi-ausgedehnte Addition für die ansteigenden Zahlen ein, die sich wesentlich von der Super-Addition unterscheidet. Andererseits ist die quasi-ausgedehnte Addition die (traditionelle) addition für reale Zahlen. Schlielich finden wir einige hnlichkeit zwischen Unendlich und 1. Die quasi-ausgedehnte Addition ist auch nützlich für Studenten, um verschiedene Arten von algebraischen Eigenschaften zu beobachten.

Keywords: exploded real numbers, didactics of mathematics

1. Introduction

In the genesis of exploded real numbers ([1], Preliminary) the Postulates and Requirements of the concept of exploded real numbers were given. We may satisfy them by the following way:

For any element ξ of the open interval (-1,1) we say that its exploded, denoted by $\overset{\sqcup}{\xi}$, is given by

$$\overset{\sqcup}{\xi} = \operatorname{area\,th} \xi = \frac{1}{2} \ln \frac{1+\xi}{1-\xi}. \tag{1.1}$$

Of course, any real number x is exploded real number, too. Because

$$x = \overset{\sqcup}{\operatorname{th} x}, \quad x \in R, \quad \operatorname{th} x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.2}$$

So, we say, that the real numbers are *visible exploded real numbers.* Otherwise, we use the symbol $\overset{\sqcup}{x}$ for every real numbers such that in the set $\overset{\sqcup}{R}$ of exploded real numbers

$$\overset{\sqcup}{x} \overset{\overset{\sqcup}{R}}{=} \overset{\sqcup}{y} \Longleftrightarrow x = y, \quad x, y \in R \tag{1.3}$$

and

$$\overset{\sqcup}{x} \overset{\overset{\sqcup}{R}}{<} \overset{\sqcup}{y} \Longleftrightarrow x < y, \quad x, y \in R. \tag{1.4}$$

As in the case $x, y \in (-1, 1)$

$$\operatorname{area\,th} x = \operatorname{area\,th} y \Longleftrightarrow x = y$$

and

$$\operatorname{area\,th} x < \operatorname{area\,th} y \Longleftrightarrow x < y$$

are fulfilled, we may use „=” and „<” for exploded real numbers, in general. By (1.3) for any exploded real number u we define its compressed $\underset{\sqcup}{u}$ by the *first inversion dentity*

$$(1.5) \qquad \overset{\sqcup}{(\underset{\sqcup}{u})} = u, \quad u \in \overset{\sqcup}{R}.$$

Of course, $\underset{\sqcup}{u}$ is a real number. Denoting $x = \underset{\sqcup}{u}$, (1.5) shows that $\overset{\sqcup}{x} = u$ so, we have the *second inversion identity*

$$(1.6) \qquad \underset{\sqcup}{(\overset{\sqcup}{x})} = x, \quad x \in R.$$

Now, for any $x \in R$, (1.2) gives

$$(1.7) \qquad \underset{\sqcup}{x} = \operatorname{th} x.$$

The $\overset{\sqcup}{1}$ and $\overset{\sqcup}{(-1)}$ are already not visible exploded numbers. Sometimes, the exploded real numbers $\overset{\sqcup}{x}$, where $x \geq 1$ or $x \leq -1$ will be mentioned as *invisible exploded real numbers*. The invisible exploded real number $\overset{\sqcup}{1}$, as the smallest exploded number which is greater than any real number is called *positive discriminator*. The invisible exploded real number $\overset{\sqcup}{(-1)}$, as the greatest exploded number which is less than any real number is called *negative discriminator*.
If we step one dimension then we can see the invisible exploded real numbers, too. For example, in the case of *complex model of exploded real numbers* ([2], Part 2. with $c = 1$) the explosion of real numbers is defined by

$$(1.8) \qquad \overset{\sqcup}{x} = (\operatorname{sgn} x)(\operatorname{area\,th}\{|x|\} + i[|x|]), \quad x \in R,$$

where $[x]$ is the gratest integer number which is less than or equal to x and $\{x\} = x - [x]$. If $x = \xi \in (-1, 1)$ then (1.8) reduces to (1.1). The equality in the complex model is understable in the usual sense of equality of complex numbers. The inequality is understable in the following sense: $\overset{\sqcup}{x} < \overset{\sqcup}{y}$ if $\operatorname{Im} \overset{\sqcup}{x} < \operatorname{Im} \overset{\sqcup}{y}$ or if $\operatorname{Im} \overset{\sqcup}{x} = \operatorname{Im} \overset{\sqcup}{y}$ then $\operatorname{Re} \overset{\sqcup}{x} < \operatorname{Re} \overset{\sqcup}{y}$. Here the positive discriminator is i, while the negative discriminator is $(-i)$. The compressed of $u \in \overset{\sqcup}{R}$ is given by

$$(1.9) \qquad \underset{\sqcup}{u} = \operatorname{Im} u + \operatorname{th} \operatorname{Re} u, \quad u \in \overset{\sqcup}{R}.$$

(See [2], (2.6) with $c = 1$.) If $u = x \in R$ then (1.9) reduces to (1.7). Returning to the set of exploded real numbers in abstract sense, we give for it an algebraic structure by the definitions

$$(1.10) \qquad u \overset{\times}{\oplus} v = \overset{\llcorner\qquad\lrcorner}{\underset{\sqcup}{u} +_{|} \underset{\sqcup}{v}}, \quad u, v \in \overset{\sqcup}{R} \quad (super - addition)$$

and

$$(1.11) \qquad u \overset{\times}{\odot} v = \overset{\llcorner\qquad\lrcorner}{\underset{\sqcup}{u} \cdot_{|} \underset{\sqcup}{v}}, \quad u, v \in \overset{\sqcup}{R} \quad (super - multiplication).$$

Clearly, $(\overset{\sqcup}{R}, \overset{\times}{\oplus}, \overset{\times}{\odot})$ is isomorphic to the ordered field $(R, +, \cdot)$.
So, they are equivalent in abstract algebraic sense, but practically they are not, because R is an open interval of $\overset{\sqcup}{R}$, only. Moreover, we have the following super-operations:

$$(1.12) \qquad u \overset{\times}{\ominus} v = \overset{\llcorner\qquad\lrcorner}{\underset{\sqcup}{u} -_{|} \underset{\sqcup}{v}}, \quad u, v \in \overset{\sqcup}{R} \quad (super - subtraction)$$

and

$$(1.13) \qquad \frac{u}{v}\!\!\!\!\!\!\sim = \overset{\llcorner\qquad\lrcorner}{\left(\frac{\underset{\sqcup}{u}}{\underset{\sqcup}{v}}\right)}, \quad u, v \in \overset{\sqcup}{R}, \quad v \neq 0, (super - division).$$

We remark that $(R, \overset{\times}{\odot})$ is a commutative semi-group, while $(R, \overset{\times}{\oplus})$ is not closed in algebraic sense. Of course, $(\overset{\sqcup}{R}, \overset{\times}{\oplus}, \overset{\times}{\odot})$ is an ordered field.

2. An intention for the extension of addition to exploded real numbers

Our intention is based on the identity

$$(2.1) \qquad \operatorname{th}(x+y) = \frac{\operatorname{th} x + \operatorname{th} y}{1 + \operatorname{th} x \cdot \operatorname{th} y}; \quad x, y \in R.$$

Let us assume, that x and y are real numbers. Together with $x+y$ they are visible exploded real numbers, too. Applying the first inversion identity (1.5) the identity (2.1) with (1.7), we can write

$$x + y = \overset{\llcorner\qquad\lrcorner}{\underline{(x+y)}} = \overset{\llcorner\qquad\qquad\lrcorner}{\left(\frac{\underset{\sqcup}{x} +_{|} \underset{\sqcup}{y}}{1 +_{|} \underset{\sqcup}{x}\, \underset{\sqcup}{y}}\right)}.$$

Using the second inversion identity (1.6) and definition (1.11) we have

$$x + y = \overset{\llcorner\qquad\qquad\lrcorner}{\left(\frac{\underset{\sqcup}{x} +_{|} \underset{\sqcup}{y}}{1 +_{|} \underline{\overset{\llcorner\quad\lrcorner}{(\underset{\sqcup}{x}\, \underset{\sqcup}{y})}}}\right)} = \overset{\llcorner\qquad\qquad\lrcorner}{\left(\frac{\underset{\sqcup}{x} +_{|} \underset{\sqcup}{y}}{1 +_{|} \underline{x \overset{\times}{\odot} y}}\right)}.$$

Applying the second inversion identity again, using that $\lfloor (\overset{\sqcup}{1}) \rfloor = 1$ by definition (1.10)

$$x+y = \left(\frac{\lfloor (\underset{\sqcup}{x} + \underset{\sqcup}{y}) \rfloor}{\lfloor \lfloor (\overset{\sqcup}{1}) \rfloor + \lfloor x \odot y \rfloor \rfloor} \right)^{\lceil\,\rceil} = \left(\frac{\lfloor x \oplus y \rfloor}{\lfloor \overset{\sqcup}{1} \oplus (x \odot y) \rfloor} \right)^{\lceil\,\rceil}$$

is obtained. Finally, definition (1.13) gives the identity

$$x+y = \frac{x \oplus y}{\overset{\sqcup}{1} \oplus (x \odot y)}; \quad x, y \in R \tag{2.2}$$

Now we controll the identity (2.2) by the complex model of exploded real numbers. Using (1.9) and (1.8) definitions (1.10), (1.11), (1.12) and (1.13) yield

$$u \oplus v =$$

$$= (\operatorname{sgn}(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u + \operatorname{Im} v + \operatorname{th} \operatorname{Re} v)) \operatorname{area\,th}\{|\operatorname{Im} u + \operatorname{th} \operatorname{Re} u + \operatorname{Im} v + \operatorname{th} \operatorname{Re} v|\} +$$

$$+ i(\operatorname{sgn}(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u + \operatorname{Im} v + \operatorname{th} \operatorname{Re} v))[|\operatorname{Im} u + \operatorname{th} \operatorname{Re} u + \operatorname{Im} v + \operatorname{th} \operatorname{Re} v|],$$

$$u \odot v =$$

$$= (\operatorname{sgn}((\operatorname{Im} u + \operatorname{th} \operatorname{Re} u)(\operatorname{Im} v + \operatorname{th} \operatorname{Re} v))) \operatorname{area\,th}\{|(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u)(\operatorname{Im} v + \operatorname{th} \operatorname{Re} v)|\} +$$

$$+ i(\operatorname{sgn}((\operatorname{Im} u + \operatorname{th} \operatorname{Re} u)(\operatorname{Im} v + \operatorname{th} \operatorname{Re} v)))[|(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u)(\operatorname{Im} v + \operatorname{th} \operatorname{Re} v)|],$$

$$u \ominus v =$$

$$= (\operatorname{sgn}(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u - \operatorname{Im} v - \operatorname{th} \operatorname{Re} v)) \operatorname{area\,th}\{|\operatorname{Im} u + \operatorname{th} \operatorname{Re} u - \operatorname{Im} v - \operatorname{th} \operatorname{Re} v|\} +$$

$$+ i(\operatorname{sgn}(\operatorname{Im} u + \operatorname{th} \operatorname{Re} u - \operatorname{Im} v - \operatorname{th} \operatorname{Re} v))[|\operatorname{Im} u + \operatorname{th} \operatorname{Re} u - \operatorname{Im} v - \operatorname{th} \operatorname{Re} v|],$$

and

$$\frac{u}{v} =$$

$$= \left(\operatorname{sgn} \frac{\operatorname{Im} u + \operatorname{th} \operatorname{Re} u}{\operatorname{Im} \nu + \operatorname{th} \operatorname{Re} \nu} \right) \operatorname{area\,th} \left\{ \left| \frac{\operatorname{Im} u + \operatorname{th} \operatorname{Re} u}{\operatorname{Im} \nu + \operatorname{th} \operatorname{Re} \nu} \right| \right\} +$$

$$+ i \left(\operatorname{sgn} \frac{\operatorname{Im} u + \operatorname{th} \operatorname{Re} u}{\operatorname{Im} v + \operatorname{th} \operatorname{Re} \nu} \right) \left[\left| \frac{\operatorname{Im} u + \operatorname{th} \operatorname{Re} u}{\operatorname{Im} \nu + \operatorname{th} \operatorname{Re} \nu} \right| \right], \quad \nu \neq 0,$$

respectively.
If u and ν are real numbers then

$$u \oplus v = (\operatorname{sgn}(\operatorname{th} u + \operatorname{th} \nu)) \operatorname{area\,th}\{|\operatorname{th} u + \operatorname{th} \nu|\} +$$

$$+ i(\operatorname{sgn}(\operatorname{th} u + \operatorname{th} \nu))[|\operatorname{th} u + \operatorname{th} v|].$$

Having that $\{|(\operatorname{th} u)(\operatorname{th} \nu)|\} = |(\operatorname{th} u)(\operatorname{th} \nu)|$ and $[|\operatorname{th} u)(\operatorname{th} \nu)|] = 0$, we obtain, that

$$u \odot v = (\operatorname{sgn}((\operatorname{th} u)(\operatorname{th} \nu)) \operatorname{area\,th} |(\operatorname{th} u)(\operatorname{th} \nu)| = \operatorname{area\,th}((\operatorname{th} u)(\operatorname{th} \nu)).$$

Having that $\operatorname{sgn}(1 + (\operatorname{th} u)(\operatorname{th} v)) = 1$, we get

$$\overset{\sqcup}{1} \oplus (u \odot \nu) = \operatorname{area\,th}\{|1 + (\operatorname{th} u)(\operatorname{th} \nu)|\} + i[|1 + (\operatorname{th} u)(\operatorname{th} \nu)|].$$

Using the identity (2.1) we can write

$$\frac{\operatorname{Im}(u \oplus \nu) + \operatorname{th} \operatorname{Re}(u \oplus \nu)}{\operatorname{Im}(\overset{\sqcup}{1} \oplus (u \odot v)) + \operatorname{th} \operatorname{Re}(\overset{\sqcup}{1} \oplus (u \odot \nu))} =$$

$$\frac{(\operatorname{sgn}(\operatorname{th} u + \operatorname{th} \nu))([|\operatorname{th} u + \operatorname{th} \nu|] + \{|\operatorname{th} u + \operatorname{th} \nu|\})}{|1 + (\operatorname{th} u)(\operatorname{th} \nu)|} =$$

$$= \frac{\operatorname{th} u + \operatorname{th} \nu}{1 + \operatorname{th} \nu \cdot \operatorname{th} \nu} = \operatorname{th}(u + v), \quad u, v \in R.$$

Finally,

$$\frac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)} =$$

$$= (\operatorname{sgn} \operatorname{th}(u + \nu)) \operatorname{area\,th}\{|\operatorname{th}(u + v)|\} + i(\operatorname{sgn} \operatorname{th}(u + \nu))[|\operatorname{th}(u + v)|] =$$

$$= (\operatorname{sgn} \operatorname{th}(u + \nu)) \operatorname{area\,th} |\operatorname{th}(u + v)| = \operatorname{area\,th}(\operatorname{th}(u + \nu)) = u + \nu.$$

So, the identity (2.2) is proved another way, too.
Considering that the right hand side of (2.2), except the cases

$$\overset{\sqcup}{1} \oplus (u \odot \nu) = 0, \quad u, \nu \in \overset{\sqcup}{R} \tag{2.3}$$

has a well-defined meaning for exploded real numbers too, we define the addition of exploded real numbers by

$$u + v = \frac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)}; \quad u, v \in \overset{\sqcup}{R}; \quad 1 \oplus (u \odot v) \neq 0. \tag{2.4}$$

If the exploded numbers u and ν are satisfying the equation (2.3) we say, that they are *addition - incompetent partners*. Otherwise, u and ν are called *addition - competent* or *addition - allowed* exploded real numbers. By the equation (2.3) we can see that any exploded real number which is different from 0, has its unique addition - incompetent partner. Namely,

$$(2.5)\qquad u \quad \text{and} \quad \overline{\left(-\frac{1}{\underset{\sqcup}{u}}\right)}, \qquad (u \in \overset{\sqcup}{R}, u \neq 0)$$

are addition - incompetent partners of each others. Therefore, the operation defined under (2.4) is can not be considered a full extension of (traditional) addition. So, it is called a *quasi - extended addition.* In the case of real numbers the identity (2.2) shows that the quasi - extended addition is not different from the (traditional) addition, but it is different from the super - addition. The super - sum of real numbers may be invisible exploded real number. The 0 has not any addition - incompetent partner.
Each real number, which is different from 0, has own addition - incompetent partner among the invisible exploded real numbers. Conversely, each invisible exploded real number, which is different from the discriminators, has own addition - incompetent partner among the visible exploded real numbers. The discriminators $\overset{\sqcup}{1}$ and $\overline{(-1)}$ are addition - incompetent partners. As the compressed of exploded real numbers are always visible numbers we show this situation by the following figure:

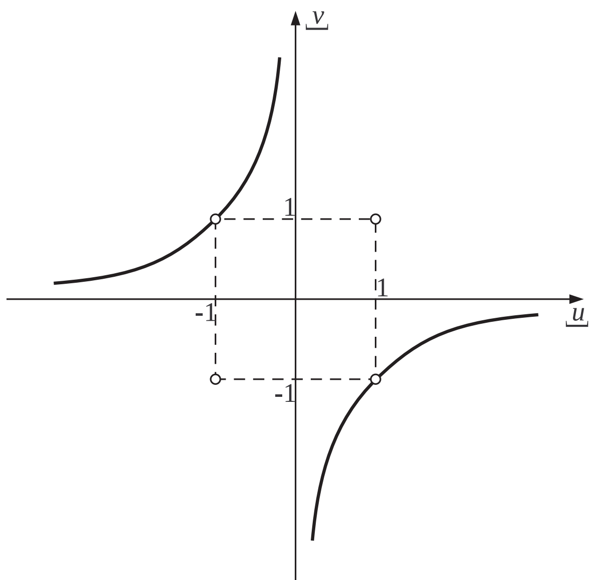

Fig.2.6

If the point $(\underset{\sqcup}{u},\underset{\sqcup}{\nu})$ is situated on the hiperbola, which is called *incompetence-hiperbol* having the equation

$$(2.7)\qquad 1 +_{|} \underset{\sqcup}{u}\,\underset{\sqcup}{\nu} = 0 \quad (u, v \in R)$$

then the exploded real numbers u and ν are addition - incompetent partners. By the complex model of exploded real numbers we are able to see both (invisible and visible) addition - incompetent partners. If we compute with quasi - extended addition in the complex model then we have to use for the quasi - extended addition another symbol, for example: #. Let us consider the following example: Let be $u = 0.5 + 2i$ which is an „invisible” exploded real number. (Here the simbol „+” does not mean the quasi - extended addition, it means the addition used in the field $(C, +, \cdot)$, where C denotes the set of complex numbers. In this case, $0.5 \# 2i \sim 0.287 + i$.) By (1.9) we have $\underset{\sqcup}{u} = 2 + \operatorname{th} 0.5 \sim 2.46$. Hence, $-\frac{1}{\underset{\sqcup}{u}} \sim -0,41$ and (1.1) with (2.5) gives that u has a visible addition - incompetent partner $\overline{\left(-\frac{1}{\underset{\sqcup}{u}}\right)} \sim -0.43$. It can be seen in the following figure:

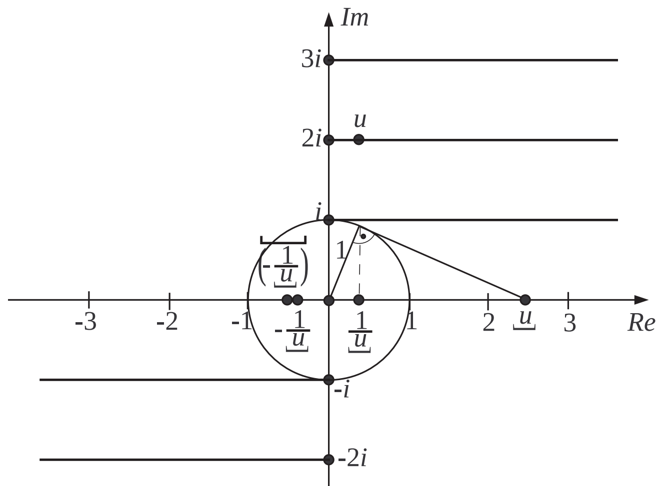

Fig.2.8

3. On the commutativity and associativity

Having the commutativity of super - addition and super - multiplication, using (2.4) by

$$u + v = \frac{u \oplus v}{\underset{\sqcup}{1} \oplus (u \odot v)} = \frac{v \oplus u}{\underset{\sqcup}{1} \oplus (v \odot u)} = v + u$$

we get

Theorem 3.1. If u and ν are addition - allowed exploded real numbers then

$$u + \nu = \nu + u$$

holds.

Our theorem concerning the associativity is a little bit complicated.

Theorem 3.2. Let be u, ν and w exploded real numbers. If they are pairwise addition - competent exploded real numbers, moreover $u + \nu$ and w are addition - allowed exploded real numbers then u and $\nu + w$ are addition - competent exploded real numbers. Moreover,

$$(u + \nu) + w = u + (\nu + w) \tag{3.3}$$

holds.

Proof of Theorem 3.2. Considering (2.3) by our assumptions we have

$$\overset{\sqcup}{1} \oplus (u \odot \nu) \neq 0 \tag{3.4}$$

$$\overset{\sqcup}{1} \oplus (\nu \odot w) \neq 0 \tag{3.5}$$

$$\overset{\sqcup}{1} \oplus (w \odot u) \neq 0 \tag{3.6}$$

and

$$\overset{\sqcup}{1} \oplus ((u+\nu) \odot w) \neq 0. \tag{3.7}$$

First, we show that u and $\nu + w$ are addition - competent exploded real numbers. Using (2.4) with (3.5) we can write

$$\overset{\sqcup}{1} \oplus \left(u \odot (v+w)\right) = \overset{\sqcup}{1} \oplus \left(u \odot \frac{v \oplus w}{\overset{\sqcup}{1} \oplus (v \odot w)}\right) =$$

$$= \frac{\overset{\sqcup}{1} \oplus (v \odot w) \oplus (u \odot v) \oplus (u \odot w)}{\overset{\sqcup}{1} \oplus (v \odot w)}.$$

On the other hand using (2.4) with (3.4) we get

$$\overset{\sqcup}{1} \oplus \left((u+v) \odot w\right) = \overset{\sqcup}{1} \oplus \left(\frac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)} \odot w\right) =$$

$$= \frac{\overset{\sqcup}{1} \oplus (u \odot v) \oplus (u \odot w) \oplus (v \odot w)}{\overset{\sqcup}{1} \oplus (u \odot v)}.$$

Hence, by (3.7) we have that

$$1 \oplus (\nu \odot w) \oplus (u \odot \nu) \oplus (u \odot w) \neq 0$$

which by (2.3) yields that u and $\nu + w$ are addition - competent exploded real numbers.

Finally, applying (2.4)

$$(u+v)+w = \frac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)} + w = \frac{\dfrac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)} \oplus w}{\overset{\sqcup}{1} \oplus \left(\dfrac{u \oplus v}{\overset{\sqcup}{1} \oplus (u \odot v)} \odot w\right)} =$$

$$= \frac{u \oplus v \oplus w \oplus (u \odot v \odot w)}{\overset{\sqcup}{1} \oplus (u \odot v) \oplus (u \odot w) \oplus (v \odot w)} =$$

$$= \frac{u \oplus \dfrac{v \oplus w}{\overset{\sqcup}{1} \oplus (v \odot w)}}{\overset{\sqcup}{1} \oplus \left(u \odot \dfrac{v \oplus w}{\overset{\sqcup}{1} \oplus (v \odot w)}\right)} = u + \frac{v \oplus w}{\overset{\sqcup}{1} \oplus (v \odot w)} = u + (v+w)$$

so, Theorem 3.2 is proved.
We remark that the necessity of assumptions (3.4) and (3.7) is clear. Observing, that condition (3.6) was not used, we have a sharper
Theorem 3.8. Let be u, v and w exploded real numbers. If u with ν and ν with w are addition - competent exploded real numbers, moreover $u+\nu$ and w are addition - allowed exploded real numbers then u and $\nu + w$ are addition - competent exploded real numbers. Moreover,

$$(u+\nu)+w = u+(\nu+w)$$

holds.
Theorem 3.8 is already sharp. The necessity of the assumption (3.5) is showed by
Example 3.9. Let us consider the exploded real numbers $u = 1$, $\nu = (-1)$ and $w = \overline{\left(\frac{e^2+1}{e^2-1}\right)}$. Clearly, $u + \nu = 0$. On the other hand, although $u+\nu$ and w are addition - competent exploded real numbers (2.3) shows that ν and w are addition - incompetent partners, because using (1.6), (1.11), (1.7), (1.2), (1.10) and (1.1)

$$\overset{\sqcup}{1} \oplus \left((-1) \odot \overline{\left(\frac{e^2+1}{e^2-1}\right)}\right) = \overset{\sqcup}{1} \oplus \overline{\left(\underline{-1} \cdot \frac{e^2+1}{e^2-1}\right)} = \overset{\sqcup}{1} \oplus \overline{\left((-th1)\frac{e^2+1}{e^2-1}\right)} =$$

$$= \overset{\sqcup}{1} \oplus \overline{\left(\frac{1-e^2}{1+e^2} \cdot \frac{e^2+1}{e^2-1}\right)} = \overset{\sqcup}{1} \oplus \overset{\sqcup}{(-1)} = \overset{\sqcup}{1+(-1)} = \overset{\sqcup}{0} = 0$$

is obtained.
The following example shows that although u, ν and w are pairwise addition - competent exploded real numbers $u+\nu$ and w are not addition - competent exploded real numbers.
Example 3.10. Let us consider the exploded real numbers $u = 1$, $\nu = 1$ and
$w = \overline{\left(-\dfrac{e^4+1}{e^4-1} \right)}$. Considering (2.3) we can see that they are pairwise addition - competent exploded real numbers, because using (1.6), (1.11), (1.7), (1.2) and (1.10)

$$\overset{\sqcup}{1} \oplus \left(1 \odot \overline{\left(-\frac{e^4+1}{e^4-1} \right)} \right) = \overset{\sqcup}{1} \oplus \overline{\underset{\sqcup}{1} \cdot \frac{e^4+1}{1-e^4}} = \overset{\sqcup}{1} \oplus \overline{(\operatorname{th} 1)\frac{e^4+1}{1-e^4}} =$$

$$= \overset{\sqcup}{1} \oplus \overline{\left(\frac{e^2-1}{e^2+1} \cdot \frac{e^4+1}{1-e^4} \right)} = \overline{\left(\frac{2e^2}{(e^2+1)^2} \right)} \neq 0$$

is obtained. On the other hand, $u+\nu$ and w are addition - incompetent partners, because using (1.6), (1.11), (1.7), (1.2), (1.10) and (1.1)

$$\overset{\sqcup}{1} \oplus ((u+\nu) \odot w) = \overset{\sqcup}{1} \oplus \left(2 \odot \overline{\left(-\frac{e^4+1}{e^4-1} \right)} \right) =$$

$$= \overset{\sqcup}{1} \oplus \overline{\underset{\sqcup}{2}\left(-\frac{e^4+1}{e^4-1} \right)} = \overset{\sqcup}{1} \oplus \overline{\left(-\frac{e^4-1}{e^4+1} \cdot \frac{e^4+1}{e^4-1} \right)} = 0$$

is obtained.

4. Idempotent and quasi-omnipotent exploded real numbers

An exploded real number u is called *idempotent, with respect to addition* if

$$u + u = u. \tag{4.1}$$

If u is a real number (visible exploded real number) then the equation (4.1) has the solution $u = 0$, only. If u may be exploded real number, then we have more solutions: Considering the left hand side of (4.1), having that for any $u \in \overset{\sqcup}{R}$

$$\overset{\sqcup}{1} \oplus (u \odot u) \neq 0$$

and using (1.10), (1.11) and (1.13) by (2.4) we have

$$u + u = \frac{u \oplus u}{\overset{\sqcup}{1} \oplus (u \odot u)} = \overline{\left(\frac{2\underset{\sqcup}{u}}{1 + (\underset{\sqcup}{u})^2} \right)}$$

while the right hand side of (4.1) by the first inversion identity (1.5) is $\overline{(\underset{\sqcup}{u})}$. Hence, the Postulate of unambiguity (see [1], Preliminary) says that

$$\frac{2\underset{\sqcup}{u}}{1+(\underset{\sqcup}{u})^2} = {}_{\llcorner}\underset{\lrcorner}{u},$$

where $\underset{\sqcup}{u}$ is already real number. Clearly, this equation has three solutions:

$$\underset{\sqcup}{u} = 0, \quad {}_{\llcorner}\underset{\lrcorner}{u} = 1 \quad \text{and} \quad \underset{\sqcup}{u} = -1.$$

So, the solutions of (4.1) are

$$u = 0, \quad u = \overset{\sqcup}{1} \quad \text{and} \quad u = \overline{(-1)}\,.$$

They are the idempotent exploded real numbers with respect to addition. An exploded real number ν is called *omnipotent, with respect to addition* if for any exploded real number u the equation

$$u + \nu = \nu \tag{4.2}$$

is fulfilled. Considering that merely the 0 has not addition - incompetent partner, but for any exploded real number u, the definition (2.4) says that

$$u + 0 = u, \tag{4.3}$$

so, omnipotent exploded real number does not exist. However, if we give a little bit milder reqirement, we may get a positive result. We say, that an exploded real number ν is called *quasi-omnipotent, with respect to addition*, if except of the addition-incompetent partner of ν, the equality (4.2) fulfills for any exploded real number u.

Casting a glance at (2.5) and having that $\nu \neq 0$ we assume that $u \neq \overline{\left(-\frac{1}{\underset{\sqcup}{\nu}}\right)}$. Considering the left hand side of (4.2), having

$$\overset{\sqcup}{1} \circledast\!\!\!\!\!\!\oplus (u \circledast\!\!\!\!\!\!\odot \nu) \neq 0$$

and using (1.10), (1.11) and (1.13) by (2.4) we have

$$u + \nu = \overline{\left(\frac{\underset{\sqcup}{u} + {}_{\llcorner}\underset{\lrcorner}{\nu}}{1 + {}_{\llcorner}\underset{\lrcorner}{u}\,\underset{\sqcup}{\nu}}\right)}$$

while the right hand side of (4.2) by the first inversion identity (1.5) is $\overset{\sqcup}{(\underset{\sqcup}{\nu})}$. Hence, the Postulate of unambiguity (see [1], Preliminary) says that

$$\frac{\underset{\sqcup}{u} \underset{\lrcorner}{+} \underset{\sqcup}{\nu}}{1 \underset{\lrcorner}{+} \underset{\sqcup}{u}\, \underset{\sqcup}{\nu}} \underset{\lrcorner}{=} \underset{\sqcup}{\nu},$$

where $\underset{\sqcup}{u}$ and $\underset{\sqcup}{\nu}$ are already real numbers. Clearly, this equation is satisfyed if

$$\underset{\sqcup}{\nu} = 1 \quad \text{and} \quad \underset{\sqcup}{\nu} = -1.$$

So, the discriminators

$$\nu = \overset{\sqcup}{1} \quad \text{and} \quad \nu = \overset{\sqcup}{(-1)}$$

satisfy our requirements. They are quasi-omnipotent exploded real numbers with respect to addition.
Having an exploded real number u, the exploded real number ν is called *inverse of* u*, with respect to addition* if u and ν are addition - competent and

(4.4) $$u + \nu = 0.$$

Of course if u is a real number, then $\nu = -u$. Considering (2.3) and (2.4) it is easy to see that, except of discriminators

$$\nu = \overset{\sqcup}{(-1)} \odot u = \overset{\sqcup}{(\underset{\sqcup}{-u})}\,; \quad u \neq \overset{\sqcup}{1},\ \overset{\sqcup}{(-1)}\,.$$

So, $\overset{\sqcup}{1}$ and $\overset{\sqcup}{(-1)}$ have no inverse with respect to addition although they are inverses of each others with respect to super-addition. (With respect to super-addition the symbols $-(\overset{\sqcup}{1}) = \overset{\sqcup}{(-1)}$ and $\overset{\sqcup}{1} = -\overset{\sqcup}{(-1)}$ are allowed, but with respect to addition are not. If we use the symbol „-" for the inverse of invisible exploded number with respect to addition, we assume that this number is different from discriminators.)
Now we are going to tell something about some similarity of countable infinity denoted by $\aleph_0$ and positive discriminator $\overset{\sqcup}{1}$. Considering the natural numbers

$$0; 1; 2; \ldots; n; \ldots \quad \text{and} \quad \aleph_0,$$

by the set - theoretical building of the natural numbers, for addition the equalities

(4.5) $$n + \aleph_0 = \aleph_0 \quad (n \in N)$$

and

(4.6) $$\aleph_0 + \aleph_0 = \aleph_0,$$

are well known. Considering the integer numbers

$$\ldots; -2; -1; 0; 1; 2; \ldots \quad \text{and} \quad \aleph_0,$$

the equality (4.5) can be extended for integer numbers:

$$u + \aleph_0 = \aleph_0 \quad (u \in Z). \tag{4.7}$$

Casting a glance at (4.2) and considering that the positive discriminator $\overset{\sqcup}{1}$ is a quasi - omnipotent exploded real number with respect to addition we have the equations

$$u + \overset{\sqcup}{1} = \overset{\sqcup}{1} \quad (u \in \overset{\sqcup}{R}, u \neq \overset{\sqcup}{(-1)}\,) \tag{4.8}$$

and

$$u + \overset{\sqcup}{(-1)} = \overset{\sqcup}{(-1)} \quad (u \in \overset{\sqcup}{R}, u \neq \overset{\sqcup}{1}\,). \tag{4.9}$$

Denoting $\overset{\sqcup}{1} = \aleph_0$ or $\overset{\sqcup}{(-1)} = \aleph_0$, observing that the equalities (4.5) and (4.6) are the special cases of (4.8) or (4.9) respectively, moreover, that $\aleph_0$ and $\overset{\sqcup}{1}$ are greater than each integer number, we can identity the countable infinity and positive discriminator with respect to addition.

5. Questions and problems

Problem 5.1 By definition (2.4) a possible concept of the sum of the sum of three exploded real numbers is the following: If u, and ν moreover, $u + \nu$ and w are addition-allowed exploded real numbers then

$$u + \nu + w = (u + \nu) + w.$$

The aim is to find a necessary and sufficient condition of *commutativity of trio* which means that the sums $u + \nu + w$, $u + w + \nu$, $\nu + u + w$, $v + w + u$, $w + u + \nu$ and $w + \nu + u$ are pairwise equal. (We remark, that if u, ν and w are pairwise addition - competent exploded real numbers, moreover the pairs $u + \nu$, w and $w + u$, ν are addition-allowed then the commutativity of trio is valid.) Introducing the sum $a + b + c + d = (a + b + c) + d$ the *commutativity of quart* has been raised, and so on...
Question 5.2 What kind of conditions are necessary or sufficient for the monotinity of quasi-extended addition? The monotonity of quasi-extended addition means that if $u < \nu$, moreover u with w, and ν and w are addition-competent exploded real numbers then $u + w < \nu + w$. (For example, a necessary condition: w is different from the quasi-omnipotent exploded real numbers. Simple sufficient condition: u, ν are positive exploded numbers and w is a positive real number.)

Question 5.3 Considering the set of exploded real numbers with operations super-addition, super-multiplication and quasi-extended addition (shortly, $(\overset{\sqcup}{R}, \overset{\times}{\underset{\times}{\oplus}}, \overset{\times}{\underset{\times}{\odot}}, +)$) we have the distributivity

$$(u \overset{\times}{\underset{\times}{\oplus}} v) \overset{\times}{\underset{\times}{\odot}} w = (u \overset{\times}{\underset{\times}{\odot}} w) \overset{\times}{\underset{\times}{\oplus}} (v \overset{\times}{\underset{\times}{\odot}} w).$$

Is there any other distributivity in $(\overset{\sqcup}{R}, \overset{\times}{\underset{\times}{\oplus}}, \overset{\times}{\underset{\times}{\odot}}, +)$?

Problem 5.4. Let us consider the set of real numbers with operations addition, multiplication and super-multiplication (shortly, $(R, +, \cdot, \overset{\times}{\underset{\times}{\odot}})$). Investigate the algebraic properties of $(R, +, \cdot, \overset{\times}{\underset{\times}{\odot}})$!

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