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ON SOME PROPERTIES OF DIFFERENTIAL OPERATOR

Maslina Darus, Rabha W. Ibrahim

Abstract. We define a differential operator for analytic functions of fractional power. A class of analytic functions containing this operator is studied. Finally, we determine conditions under which the partial sums of the linear operator of bounded turning are also of bounded turning.
Abstract. We define a differential operator for analytic functions of fractional power. A class of analytic functions containing this operator is studied. Finally, we determine conditions under which the partial sums of the linear operator of bounded turning are also of bounded turning.

Zusammenfassung. Wir definieren eine Typische analytische Funktionen der "gebrochene Macht". Es wird eine Klasse fr analytische Funktionen untersucht mit diesem Betreiber. Schlielich zeigen wir die Bedingungen, unter denen die teilweise Summen der linearen Betreiber, diese sind alle teilweise beschrnkt.

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1 Introduction

In the study of geometric function theory, great many mathematicians dated in 1900 have discussed various topics. The famous Bieberbach conjecture [1] in 1916 had given tremendous impact in the study of analytic univalent functions. The credit should be given to Koebe [2] who in 1907 stated that for $f \in S$, and $f(z) = z + a_2z^2 + a_3z^3...$, then $|a_n| \leq c$, c is a constant. This was a curiosity provoking discovery and attracted many leading mathematicians including Bieberbach. Bieberbach [1] stated that $|a_n| \leq n$ for all $n \geq 2$ with equality for the Koebe function k, defined by $k(z) = \frac{z}{(1-z)^2}$ for z in a unit disc $U = \{z : |z| < 1\}$. Though Louis de Brange [3] in 1984 solved the conjecture, many interesting problems remain.

However, as time goes by, the normalised analityic univalent functions are defined by operators noted as differential or integral operators. Now it has become a trend in the research area. Perhaps, Ruscheweyh [4] was the pioneer in the differential operator who introduced it in 1975. It followed by Salagean [5] in 1983 giving another version of differential operator. In the same paper, he introduced an integral operator. Many properties have been discussed and studied by many researchers for these two operators. Recently Shaqsi and Darus [6] studied both differential of Ruscheweyh and Salagean, and generalised them by using polylogarithms functions. These operators motivate us to create another type of differential operator and obtain certain conditions on bounded turning. In addition, we obtain the Cesaro means for the operator defined. Now let us define the operator as the following:

Let Σ_α denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}, \ \Big(\alpha \geq 1\Big), \tag{1}$$

which are analytic in the unit disk $U := \{z \in \mathbb{C}, |z| < 1\}$. For the Hadamard product or convolution of two power series $f(z)$ defined in (1) and a function $g(z)$ where $g(z) = z + \sum_{n=2}^{\infty} b_n z^{n+\alpha-1}, \ (z \in U)$ is

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^{n+\alpha-1}, \ (z \in U).$$

We define a differential operator as follows

$$\begin{aligned} D^0 f(z) &= f(z) = z + \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}, \\ D^1 f(z) &= f(z) - z(1 - f'(z)) = z + \sum_{n=2}^{\infty} (n+\alpha) a_n z^{n+\alpha-1} \\ D^2 f(z) &= D\Big(Df(z)\Big) = z + \sum_{n=2}^{\infty} (n+\alpha)^2 a_n z^{n+\alpha-1} \\ &\vdots \\ D^k_\alpha f(z) &= D\Big(D^{k-1} f(z)\Big) = z + \sum_{n=2}^{\infty} (n+\alpha)^k a_n z^{n+\alpha-1}. \end{aligned} \tag{2}$$

It is clear that $D^k_\alpha f(z) \in \Sigma_\alpha$ and $D^k_\alpha f(0) = 0$ and $[D^k_\alpha f(0)]' = 1$.

Other analytic classes of fractional power are studied by the authors as well (see [7-9]).

For $0 \leq \mu < 1$, let $\Theta_\alpha(\mu)$ denote the class of functions f of the form (1) so that $\Re\{f'\} > \mu$ in U. The functions in $\Theta_\alpha(\mu)$ are called functions of bounded turning such that the functions in $\Theta_\alpha(\mu)$ are univalent and also close-to-convex in U (see [10]).
The $m-th$ partial sums $F_m(z)$ of the linear operator (2) are given by

$$F_m(z) = z + \sum_{n=2}^{m} (n+\alpha)^k a_n z^{n+\alpha-1}, \ (z \in U). \tag{3}$$

We show that if f of the form (1) and belongs to the class $\Theta_\alpha(\mu)$, i.e. $\Re\{f'(z)\} > \mu$ then $F_m(z)$ are also belong to the class $\Theta_\alpha(\mu)$, i.e. $\Re\{F'_m\} > \mu, \ m = 1, 2, \ldots$. For this purpose, we need to the following preliminaries in the sequel.

Lemma 1.1. [11] For $z \in U$ we have

$$\Re\Big\{\sum_{n=1}^{j} \frac{z^n}{n+2}\Big\} \geq -\frac{1}{3}. \tag{4}$$

Lemma 1.2. [10, Vol. I] Let $P(z)$ be analytic in U, such that $P(0) = 1$, and $\Re(P(z)) > \frac{1}{2}$ in U. For functions Q analytic in U the convolution function $P * Q$ takes values in the convex hull of the image on U under Q.

2 Main Results

By making use Lemma 1.1 and Lemma 1.2, we illustrate the conditions under which the $m-$th partial sums (3) of the functions in Σ_α of bounded turning are also of bounded turning.

Theorem 2.1. Let $f \in \Sigma_\alpha$. If $\frac{1}{2} < \mu < 1$ and $f(z) \in \Theta_\alpha(\mu)$, then $F_m(z) \in \Theta_\alpha\big(\frac{2+\mu}{3}\big)$.

Proof. Let f be of the form (1) and $f \in \Theta_\alpha(\mu)$ that is

$$\Re[f'(z)] > \mu, \;\; (\frac{1}{2} < \mu < 1).$$

Implies

$$\Re\Big\{1 + \sum_{n=2}^{\infty} a_n(n+\alpha-1)z^{n+\alpha-2}\Big\} > \mu > \frac{1}{2}.$$

Now for $\frac{1}{2} < \mu < 1$ we have

$$\Re\Big\{1 + \sum_{n=2}^{\infty} a_n \frac{(n+\alpha-1)}{(1-\mu)} z^{n+\alpha-2}\Big\} > \Re\Big\{1 + \sum_{n=2}^{\infty} a_n(n+\alpha-1)z^{n+\alpha-2}\Big\}$$

then

$$\Re\Big\{1 + \sum_{n=2}^{\infty} a_n \frac{(n+\alpha)^k(n+\alpha-1)}{(1-\mu)} z^{n+\alpha-2}\Big\} > \frac{1}{2}. \tag{5}$$

Applying the convolution properties of power series to $F'_m(z)$ we may write

$$\begin{aligned} F'_m(z) &= 1 + \sum_{n=2}^{m}(n+\alpha)^k(n+\alpha-1)a_n z^{n+\alpha-2}\Big\} \\ &= \Big[1 + \sum_{n=2}^{m} \frac{(n+\alpha)^k(n+\alpha-1)}{(1-\mu)} a_n z^{n+\alpha-2}\Big] * \Big[1 + \sum_{n=2}^{m}(1-\mu)z^{n+\alpha-2}\Big] \\ &:= P(z) * Q(z). \end{aligned} \tag{6}$$

From Lemma 1.1 for $j = m-1$ and $\alpha \geq 1$, we obtain

$$\Re\Big\{\sum_{n=2}^{m} \frac{z^{n-1+\alpha-1}}{n+1}\Big\} \geq \Re\Big\{\sum_{n=2}^{m} \frac{z^{n-1}}{n+1}\Big\} \geq -\frac{1}{3}. \tag{7}$$

Since

$$\Re\Big\{\sum_{n=2}^{m} z^{n+\alpha-2}\Big\} \geq \Re\Big\{\sum_{n=2}^{m} \frac{z^{n+\alpha-2}}{n+1}\Big\}. \tag{8}$$

Then yields

$$\Re\Big\{\sum_{n=2}^{m} z^{n+\alpha-2}\Big\} \geq -\frac{1}{3}. \tag{9}$$

Thus a computation gives

$$\Re\Big\{Q(z)\Big\} = \Re\Big\{1 + \sum_{n=2}^{m}(1-\mu)z^{n+\alpha-2}\Big\} > \frac{2+\mu}{3}.$$

On the other hand, the power series

$$P(z) = \Big[1 + \sum_{n=2}^{m} \frac{(n+\alpha)^k(n+\alpha-1)}{(1-\mu)} a_n z^{n+\alpha-2}\Big], \;\; (z \in U)$$

satisfies: $P(0)=1$ and

$$\Re\Big\{P(z)\Big\} = \Re\Big[1+\sum_{n=2}^{m}\frac{(n+\alpha)^k(n+\alpha-1)}{(1-\mu)}a_n z^{n+\alpha-2}\Big] > \frac{1}{2}, \ \ (z\in U).$$

Therefore, by Lemma 1.2, we have

$$\Re\Big\{F'_m(z)\Big\} > \frac{2+\mu}{3}, \ \ (z\in U).$$

This concludes the Main Theorem.

From the partial sum

$$s_m(z) = z+\sum_{n=2}^{m} a_n z^{n+\alpha-1}, \ \ z\in U,$$

with $s_1(z)=z$ we construct the Cesáro means $\sigma_m(z)$ of $f\in\Sigma_\alpha$ by

$$\begin{aligned}
\sigma_m(z,f) &= \frac{1}{m}\sum_{n=1}^{m} s_n(z)\\
&= \frac{1}{m}\big[s_1(z)+..+s_m(z)\big]\\
&= \frac{1}{m}\big[z+(z+a_2z^{\alpha+1})+..+(z+...+a_mz^{\alpha+m-1})\big]\\
&= \frac{1}{m}\big[mz+(m-1)a_2z^{\alpha+1}+..+a_mz^{\alpha+m-1})\big]\\
&= z+\sum_{n=2}^{m}(\frac{m-n+1}{m})a_nz^{\alpha+n-1}\\
&= f(z)*[z+\sum_{n=2}^{m}(\frac{m-n+1}{m})z^{n+\alpha-1}]\\
&= f(z)*g_m(z)
\end{aligned}$$

where

$$g_m = z+\sum_{n=2}^{m}(\frac{m-n+1}{m})z^{n+\alpha-1}. \tag{10}$$

Now, we have the following result:

Theorem 2.2. Let $f\in\Sigma_\alpha$. If $\frac{1}{2}<\mu<1$ and $f(z)\in\Theta_\alpha(\mu)$, then $\sigma_m(z,D^k_\alpha f(z))\in\Theta_\alpha\big(\frac{3m+\mu-1}{3}\big)$.

Proof. By using the same method in Theorem 2.1, we obtain

$$\begin{aligned}
[\sigma_m(z,D^k_\alpha f(z))]'(z) &= 1+\sum_{n=2}^{m}(n+\alpha)^k(n+\alpha-1)(\frac{k-n+1}{k})a_nz^{n+\alpha-2}\\
&= \Big[1+\sum_{n=2}^{m}\frac{(n+\alpha)^k(n+\alpha-1)}{(1-\mu)}a_nz^{n+\alpha-2}\Big]\\
&\quad *\Big[1+\sum_{n=2}^{m}(1-\mu)(\frac{m-n+1}{m})z^{n+\alpha-2}\Big]\\
&:= P(z)*Q(z).
\end{aligned} \tag{11}$$

Thus as $n \to m$ a small computation gives

$$\Re\Big\{Q(z)\Big\} = \Re\Big\{1 + \sum_{n=2}^{m} \frac{(1-\mu)}{m} z^{n+\alpha-2}\Big\} > \frac{3m+\mu-1}{3}.$$

Hence the proof.

Note that in [12], the authors determined the Cesáro means for operators containing Fox-Wright functions for functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ (z \in U).$$

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Authors

Maslina Darus, School of Mathematical Sciences, Faculty of science and Technology, Universiti Kebangsaan Malaysia, Bangi, Malaysia
maslina@ukm.my
Rabha W. Ibrahim, School of Mathematical Sciences, Faculty of science and Technology, Universiti Kebangsaan Malaysia, Bangi, Malaysia
rabhaibrahim@yahoo.com

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