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THE SUPER-CONE

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Abstract: Using the concept of exploded and compressed numbers the author constructs the super-cone which is able to turn upon the border of three dimensional space and breaks through it. The introduction of super-cone gives a possibility for students to see the properties of traditional cone while the super-cone is not a traditional cone. Moreover we show that an unbounded super-cone is a proper subset of an unbounded super-paraboloid such that they have the same infinitely large highness.

Key Words: explosion and compression of real numbers, super-operations: addition, multiplication, extraction and division, super square-root, super-cone, super-shift transformation, super paraboloid

1. Preliminary and Notations

The concept of exploded real numbers and the super-operations form the basis of our calculations.

The postulates and requirements of the concept of exploded real numbers were given in [1]. We may satisfy them in the following way:

The exploded of the $u \in R$ is given by

$$\overline{u} = (\operatorname{sgn} u)\left(\frac{1}{2}\ln\frac{1+\{|u|\}}{1-\{|u|\}} + i[|u|]\right),$$

where $[x]$ is the greatest integer number which is less than or equal $x \in R$ and $\{x\} = x - [x]$. So, the set of exploded numbers $\overline{R}$ is a proper subset of complex numbers. This model of exploded numbers was introduced by Szalay in [2].

If u is an element of the open interval (-1,1) then:

$$\overline{u} = area\,th\,u = \frac{1}{2}\cdot\ln\frac{1+u}{1-u} \tag{1.1}$$

Of course, any real number x is exploded real number, too, given by the formula:

$$x = \overline{th\,x} \qquad x \in R \qquad th\,x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \tag{1.2}$$

For any exploded real number u we define its compressed $\underline{u}$ by the first inversion identity:

$$\overline{(\underline{u})} = u \qquad u \in \overline{R} \tag{1.3}$$

Denoting $x = \underline{u}$, (1.3) shows that $\overline{x} = u$, and we have the second inversion identity:

$$\underline{(\overline{x})} = x \qquad x \in R \tag{1.4}$$

Using the above mentioned identities, the (1.2) gives:

$$\underline{x} = th\,x \qquad \text{for any} \qquad x \in R \tag{1.5}$$

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The set of super-operation used to obtain the super-cone equation are:

$$u +^s v = \overline{\underline{u} + \underline{v}} \qquad u, v \in \overline{R} \qquad \text{(super-addition)}, \tag{1.6}$$

$$u -^s v = \overline{\underline{u} - \underline{v}} \qquad u, v \in \overline{R} \qquad \text{(super-substraction)}, \tag{1.7}$$

$$u \circ^s v = \overline{\underline{u} \cdot \underline{v}} \qquad u, v \in \overline{R} \qquad \text{(super-multiplication)}, \tag{1.8}$$

$$u /^s v = \overline{\left(\frac{\underline{u}}{\underline{v}}\right)} \qquad u, v \in \overline{R} \qquad \text{(super-division)}, \tag{1.9}$$

$$\sqrt{u}^{\,s} = \overline{\sqrt{\underline{u}}} \qquad u \in \overline{R} \qquad \text{(super-square root)}. \tag{1.10}$$

An ordered algebraic structure for the set $\overline{R}$ by the super-operations was given in [3].

The familiar three dimensional space

$$R^3 = \{(u, v, w) := \overline{-1} < u < \overline{1};\ \overline{-1} < v < \overline{1};\ \overline{-1} < w < \overline{1}\},$$

with the rectangular coordinate system u, v, w is an open cube in the exploded three dimensional space

$$\overline{R^3} = \{(u, v, w) : u, v, w \in \overline{R}\}$$

Considering a set $H \subseteq \overline{R^3}$, the subset

$$H_{box} = H \cap R^3$$

is called the box-phenomenon of H. It is possible that a box-phenomenon is empty. Clearly, $\overline{R^3}_{box} = R^3$. Moreover, if $H \subseteq R^3$ then $H_{box} = H$.

2. Super-Cones with Invariable Bases

The aim is to construct a super-cone which is able to turn upon the familiar three dimensional space using the exploded numbers and the super-operations.

Having a parameter γ such that $0 < \gamma < \overline{1}$ we consider the super-circle defined by

$$C = \{(u, v, 0) : \sqrt{(u \circ^s u) +^s (v \circ^s v)}^{\,s} \le \gamma;\quad u, v \in \overline{R}\}$$

as the base of the super-cone. This base is situated on the super-plane $\overline{R}^2 = \{(u, v, 0) : u, v \in \overline{R}\}$. We can see $|u| \le \gamma;\ |v| \le \gamma$, so u and v are real numbers. This implies that $C \subset R^2 = \{(u, v, 0) : u, v \in R\}$.

Now, we fix $\gamma = 1$. Using (1.1), (1.6), (1.8) and (1.10) we have that the border-curve of the base C has the equation

$$area\ th\left(\sqrt{(th\ u)^2 + (th\ v)^2}\right) = 1$$

and it can be seen in Figure 1.

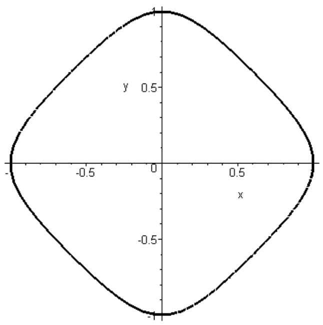

Figure 1.

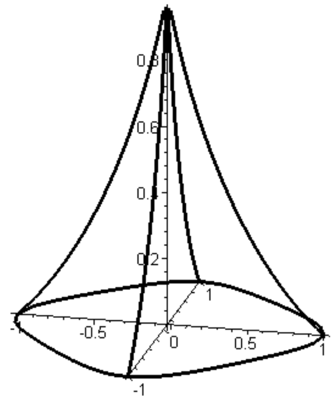

Figure 2

We can define the super-cone, using the second parameter $\mu \in \overline{R}$ such that $0 < \mu$

$$\Lambda = \left\{ (u, v, w) \in \overline{R^3} : 0 \le w \le \mu \circ^s \left(\bar{1} -^s \sqrt{\left(u \circ^s u\right) +^s \left(v \circ^s v\right)}^{\,s} /^s \gamma \right);\ 0 < \mu;\ \mu \in \overline{R};\ (u, v) \in C \right\}$$

The lower border-surface is the base C and the upper border-surface has the equation:

$$w = \mu \circ^s \left(\bar{1} -^s \sqrt{\left(u \circ^s u\right) +^s \left(v \circ^s v\right)}^{\,s} /^s \gamma \right) \tag{2.1}$$

Using the inversion identities (1.3) and (1.4) and the super-operations the right hand side of the equation (2.1) can be expressed by familiar operations:

$$w = \overline{\left(\underline{\mu} \cdot \left(1 - \frac{\sqrt{(\underline{u})^2 + (\underline{v})^2}}{\underline{\gamma}} \right) \right)} \tag{2.2}$$

Let us assume that $(u, v) \in C$ and $\mu \in R$. Using (1.5) and (1.1) the mathematical form of (2.2) is

$$w = areath\left(th\mu \cdot \left(1 - \frac{\sqrt{th^2 u + th^2 v}}{th\gamma} \right) \right).$$

Now we investigate the super-cone Λ in the following four cases:

Case I. $\gamma = 1$ and $\mu = 1$. In this case $\Lambda_{box} = \Lambda$.

The super-cone equation (2.2) has the form:

$$w = area\ th\left((th\,1) \cdot \left(1 - \frac{\sqrt{(th\,u)^2 + (th\,v)^2}}{th\,1} \right) \right)$$

with $\sqrt{(th\,u)^2 + (th\,v)^2} \le th\,1$

(2.3)

Super-cone Λ having the base C and highness 1 can be seen in Figure 2.

Case II. $\gamma = 1$ and $\mu = 2$. In this case $\Lambda_{box} = \Lambda$.

Using (1.1) and (1.5) the equation (2.2) has the form

$$w = area\ th\left((th\ 2) \cdot \left(1 - \frac{\sqrt{(th\,u)^2 + (th\,v)^2}}{th\,1} \right) \right)$$

with $\sqrt{(th\,u)^2 + (th\,v)^2} \leq th\,1$ (2.4)

Super-cone Λ having the base C and highness 2 can be seen in Figure 3.

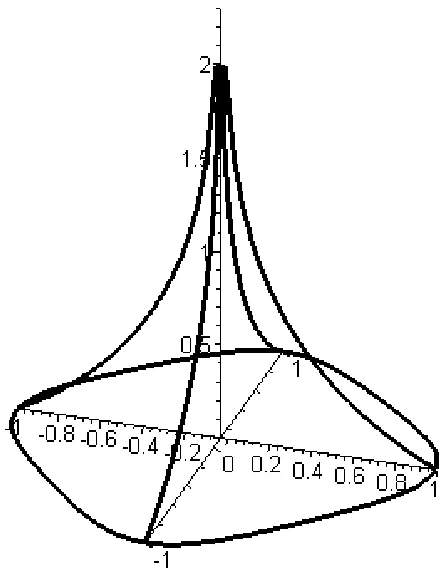

Figure 3.

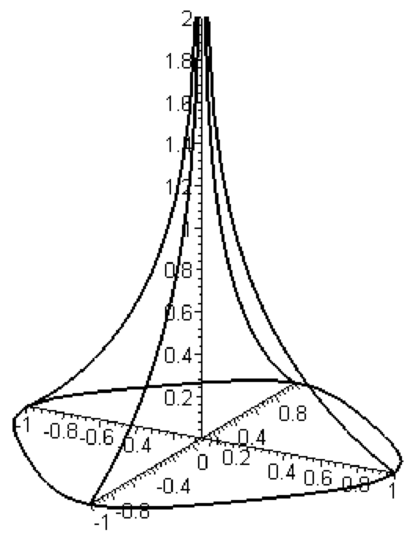

Figure 4.

Case III. $\gamma = 1$ and $\mu = \bar{1}$. In this case $\Lambda_{box} \neq \Lambda$, because the peak point $(0,0,\bar{1}) \in \Lambda$ but $(0,0,\bar{1}) \notin R^3$. Only the peak point doesn't belong to the R^3, it is situated on the „upper" border of the three dimensional space R^3, so it is invisible. By (1.1), (1.4), (1.5) and (2.2) the equation of upper border-surface of Λ_{box} has the form

$$w = area\ th\left(1 \cdot \left(1 - \frac{\sqrt{(th\,u)^2 + (th\,v)^2}}{th\,1} \right) \right)$$

with $\sqrt{(th\,u)^2 + (th\,v)^2} \leq th\,1$ and $(u,v) \neq (0,0)$. (2.5)

The box-phenomenon of super-cone having the base C and highness $\bar{1}$ can be seen in Figure 4.

Let us compare the super-cone mentioned above with the super-paraboloid having highness $\bar{1}$ and the same base C.

Considering the super-paraboloid (see Szalay [4])

$$P=\left\{(u,v,w)\in\overline{R^3}:0\le w\le\bar{1}-^{s}\left(\overline{\left(\frac{1}{th^2 1}\right)}\circ^{s}\left((u\circ^{s}u)+^{s}(v\circ^{s}v)\right)\right);(u,v)\in C\right\}$$

the equation of upper border-surface of P_{box} has the form

$$w=areath\left(1-\frac{th^2u+th^2v}{th^2 1}\right)\text{ with }\sqrt{th^2u+th^2v}\le th\,1\text{ and }(u,v)\ne(0,0).$$

Because

$$\frac{th^2u+th^2v}{th^2 1}\le\frac{\sqrt{th^2u+th^2v}}{th1};(u,v)\in C$$

$\Lambda^{box}\subset P^{box}$ is obtained, although they have the same base C and highness $\mu=\bar{1}$. The super-paraboloid with base C and highness $\bar{1}$ can be seen in Figure 5.

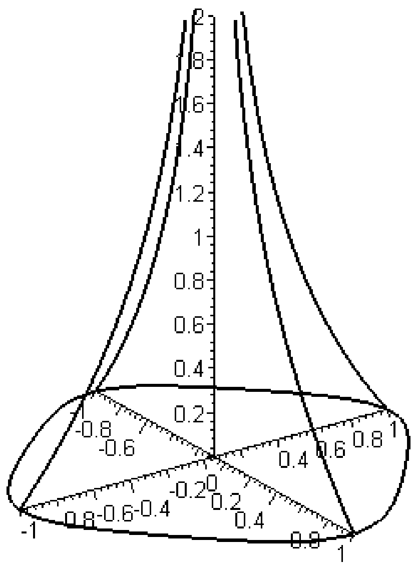

Figure 5.

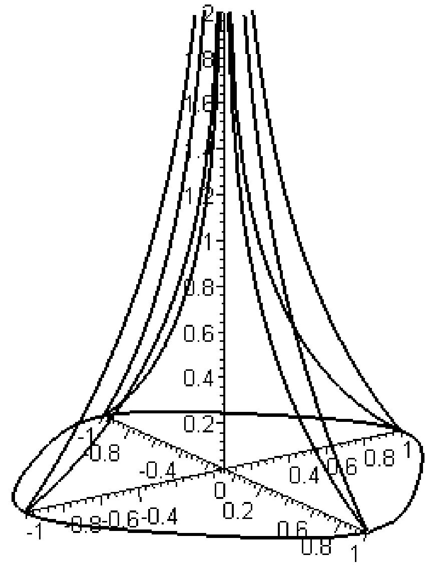

Figure 6.

Figure 6. shows the super-cone and the super-paraboloid, we can see the box-phenomenon of the super-cone is inside the box-phenomenon of the super-paraboloid.

Case IV. $\gamma=1$ and $\mu=\overline{\left(\frac{3}{2}\right)}$. In this case the peak point $\left(0,0,\overline{\left(\frac{3}{2}\right)}\right)$ is „higher" than „upper" border of the three dimensional space R^3. Many points, including the peak point $\left(0,0,\overline{\left(\frac{3}{2}\right)}\right)\in\Lambda$, but $\notin R^3$, therefore in this case $\Lambda_{box}\ne\Lambda$. If we consider (2.2) with $w=\bar{1}$; $\underline{\mu}=\frac{3}{2}$; $\underline{\gamma}=th1$, at the highness $\bar{1}$ (the „upper" border of the three-dimensional space R^3) the level curve has the equation:

$$\bar{1} = \overline{\left(\frac{3}{2}\right)\cdot\left(1-\frac{\sqrt{(\underline{u})^2+(\underline{v})^2}}{th1}\right)}$$

which by (1.4) and (1.5) has the simpler form

$$\sqrt{(th\,u)^2+(th\,v)^2} = \frac{th1}{3} \tag{2.6}$$

We can see, that over the domain

$$D=\left\{(u,v,0):\sqrt{(th\,u)^2+(th\,v)^2} \le \frac{th1}{3} \approx 0{,}25\right\}$$

Λ_{box} has not bound in the three dimensional space. The domain D can bee seen in Figure 7.

By (1.1), (1.4), (1.5) and (2.6) the equation of upper border-surface of Λ_{box} has the form

$$w = area\,th\left(\frac{3}{2}\cdot\left(1-\frac{\sqrt{(th\,u)^2+(th\,v)^2}}{th\,1}\right)\right)$$

The box-phenomenon of super-cone Λ having the base C and highness $\overline{\left(\frac{3}{2}\right)}$ can be seen in Figure 8.

It is important to see that the „corridor":

$K=\{(u,v,w)\in R^3 : (u,v)\in D;\ 0\le w\le \bar{1}\}$ is not empty, its points, with the exception of points $(u,v,\bar{1})$, with $\sqrt{(th\,u)^2+(th\,v)^2} = \frac{th1}{3}$ belong to Λ_{box} .

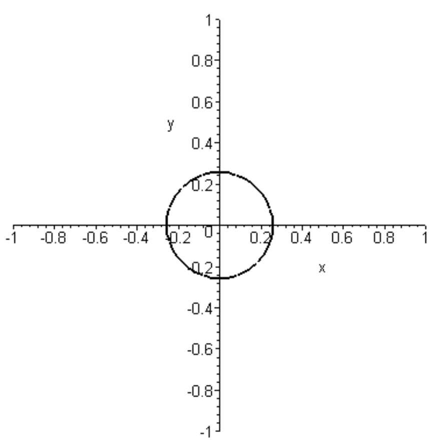

Figure 7

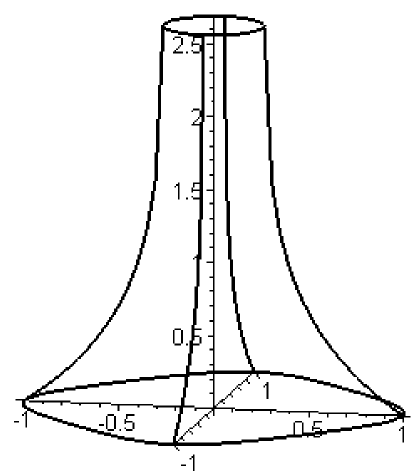

Figure 8.

3. Applications of Special Super-Shift Transformation

I. The object is to see the invisible part of super-cone with parameters $\gamma = 1$ and $\mu = \overline{1{,}5}$. We want to see what is the continuation of „corridor" K, „over" the „upper" border of three dimensional space R^3 like? To answer this question we need to use the following super-shift transformation:

Let us consider the point $O_\infty = \left(0,0,\bar{1}\right) \in R^3$ the transformation

$$\xi = u \qquad \eta = v \qquad \zeta = w -^s \bar{1} \qquad (u,v,w) \in \overline{R^3} \tag{3.1}$$

is called a special super-shift transformation generated by O_∞. This super-shift transformation moves the rectangular Descartes coordinate-system, having the axes „u" , „v" and „w" with origo $O = (0,0,0)$ into a new system, having the axes „ξ" , „η" and „ζ" with origo $O_\infty = \left(0,0,\bar{1}\right)$. (In the new system the point $O_\infty = \left(0,0,\bar{1}\right)$ has the coordinates $\xi = 0$; $\eta = 0$; $\zeta = 0$, while the point $O = (0,0,0)$ has the coordinates $\xi = 0$; $\eta = 0$; $\zeta = 0 -^s \bar{1} = \overline{0-1} = \overline{(-1)}$.) In this new system we can see an other three dimensional part

$$\Theta^3 = \left\{(u,v,w) : \overline{-1} < u < \bar{1}\,;\, \overline{-1} < v < \bar{1}\,;\, 0 < w < \bar{2}\right\}$$

of the exploded three dimensional space $\overline{R^3}$, which is different from our traditional three dimensional space R^3. More precisely,

$$R^3 \cap \Theta^3 = \left\{(u,v,w) : \overline{-1} < u < \bar{1}\,;\, \overline{-1} < v < \bar{1}\,;\, 0 < w < \bar{1}\right\}$$

and

$$\Theta^3 \setminus R^3 = \left\{(u,v,w) : \overline{-1} < u < \bar{1}\,;\, \overline{-1} < v < \bar{1}\,;\, \bar{1} < w < \bar{2}\right\}$$

The invisible part of super-cone Λ with parameters $\gamma = 1$ and $\mu = \overline{1{,}5}$, is a subset of $\Theta^3 \setminus R^3$. Moreover, the common part of the invisible part of super-cone Λ with parameters $\gamma = 1$ and $\mu = \overline{1{,}5}$ and the „upper" border of traditional three dimensional space R^3 is the domain

$$D_{upper} = \left\{\left(\xi,\eta,\bar{1}\right) : \sqrt{(th\ \xi)^2 + (th\ \eta)^2} \le \frac{th\ 1}{3}\right\}$$

The domain D_{upper} is the base of the invisible part of the super-cone Λ with parameters $\gamma = 1$ and $\mu = \overline{1{,}5}$.

Using (3.1) by (2.2) we can get the equation of the upper border surface of the continuation of „corridor" K, „over" the „upper" border of three dimensional space R^3 :

$$\zeta +^s \bar{1} = \overline{\frac{3}{2}\cdot\left(1 - \left(\frac{\sqrt{(th\ \xi)^2 + (th\ \eta)^2}}{th\ 1}\right)\right)}$$

which has the simpler form

$$\zeta = area\ th\left(\frac{1}{2} - \frac{3}{2}\cdot\left(\frac{\sqrt{(th\ \xi)^2 + (th\ \eta)^2}}{th\ 1}\right)\right) \qquad \text{with} \quad \sqrt{(th\,u)^2 + (th\,v)^2} \le \frac{th1}{3}\ .$$

By (3.1) we can controll, that the peak point of super-cone Λ is the point $\left(0,0, area\,th\frac{1}{2} +^{s} \bar{1}\right)$ and that is $\left(0,0,\overline{1{,}5}\right)$. The invisible part of super-cone Λ with parameters $\gamma = 1$ and $\mu = \overline{1{,}5}$ can be seen in Figure 9.

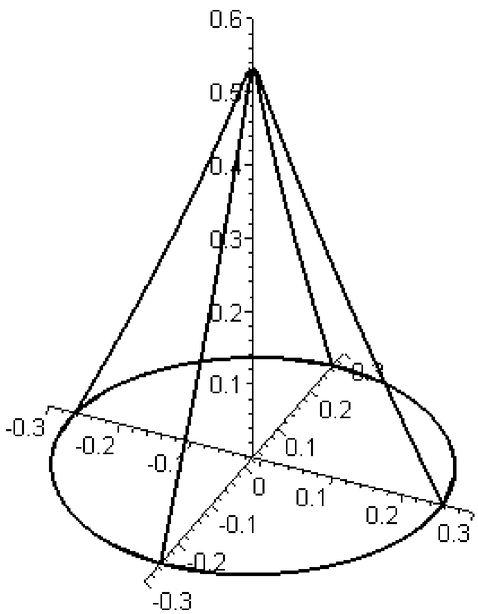

Figure 9.

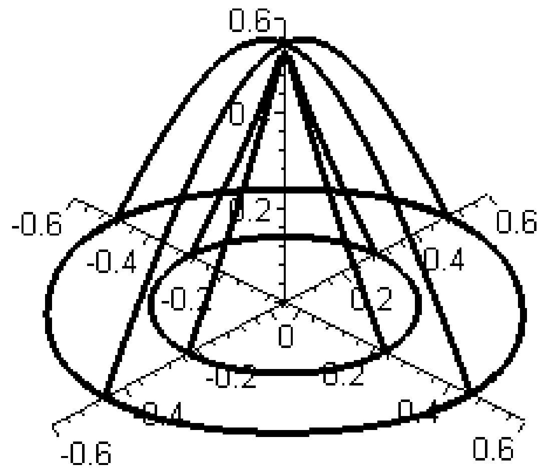

Figure 10.

It is also interesting to see the invisible part of the super-cone and the invisible part of the super-paraboloid having highness $\overline{1{,}5}$ on the same figure. Figure 10 shows that the invisible part of the super-cone is inside the invisible part of the super-paraboloid, and the upper border-surface of the super-paraboloid is bigger than the one of the super-cone.

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