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MATHEMATICAL INDUCTION AND RECURSIVE DEFINITION IN TEACHING TRAINING

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Abstract. The method of proof by mathematical induction follows from Peano axiom 5. We give three properties which are often used in the proofs by mathematical induction. We show that these are equivalent. Supposing the well-ordering property we prove the validity of this method without using Peano axiom 5. Finally, we introduce the simplest form of recursive definition.

Keywords: mathematical induction, Peano axioms, well-ordering property, recursive definition.

Introduction

In primary and secondary schools, one of the main tasks of mathematical teaching is to develop the concept of numbers. Our students who will be the future teachers have some knowledge of the concept of numbers. Our aim is to extend and deepen it. If we want our students to have more knowledge of natural numbers and their structures, then we have to introduce Peano axioms. An essential part of Peano axioms is the postulate of mathematical induction (Peano axiom 5), which is not the same as the method of proof by mathematical induction. This method of proof is the consequence of Peano axiom 5.The material we are studying in this article can be used in secondary schools and teacher training.

In the history of mathematics first Francesco Maurolicio used the method of proof by mathematical induction in his book, Arithmeticorum libri fuo (1575). In it he proved that the sum of the first n positive odd integers is n^2 (see [1]). The secondary school pupils are studying the method of proof by mathematical induction [2], but they don't know that it follows from the Peano axiom 5, or that it can be proved without using it. They don't understand it, and they use it automatically in divisibility problems, certain geometrical problems and others.

Mathematical induction

We denote the set $\{0,1,...\}$ of all natural numbers by $\mathbb{N}$ .The Peano axiom 5 is as follows:

The natural number 0 has the property T, and if the natural number k has the property T, then its successor, the natural number k+1 has also the property T, and it is true for every k. Then every natural number n has the property T.

Remark. In Peano axiom 5 we don't use the notions of set theory, but we often need the following form of Peano axiom 5:

Denote a subset of $\mathbb{N}$ which contains 0, and which contains n+1, whenever it contains n by A, then A=$\mathbb{N}$. Let P(n) be a statement depending on the natural number n. From the Peano axiom 5 we get the following:

Theorem 1. If

(a) P(0) is true, and

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(b) if P(k) is true, then P(k+1) is true

for every natural number k;

then P(n) is true for every natural number n.

Proof. Let the property T be the following:

The natural number n has the property T if only and if the statement P(n) is true.

From (a) we get that 0 has the property T. By (b) we have the following: If the natural number k has the property T, then the natural number k+1 also has the property T , and it is true for every k. From Peano axiom 5 follows that every natural number n has the property T. Therefore P(n)is true for every natural number n.

We often use the following three properties in proofs by mathematical induction (see [3]):

(I) If

(a) P(0) is true, and

(b) if P(k) is true, then P(k+1) is true

for every natural number k;

then P(n) is true for every natural number n.

(II) If

(a) P(0) is true, and

(b) if P(0), P(1) ,…, P(k) are true, then P(k+1) is true

for every natural number k;

then P(n) is true for every natural number n.

(III) If

(a) P(0) is true, and

(b) if P(k) is false, then there exists l with l<k for which P(l) is also false, and it is true for every natural number k;

then P(n) is true for every natural number n.

Remark. 1. The property (III) is called the method of infinite descent.

2. There are several useful illustrations of the above mentioned properties. One of these involves an infinite row of stairs where each stair is labelled starting with 0, 1,…,and so on. Now let P(n) be the following statement:

We can step up stair n.

Then the property (I) is the following:

Suppose that (I)(a) and (I)(b) are true. These mean that we can step up stair 0, and if we can step up stair k, then we can step up stair k+1, and it is true for every k. Thus using (I)(a) we can step up stair 0, and using (I)(b) we can step from stair 0 up stair 1, then also using (I)(b) we can step from stair 1 up to step 2,…, and so on. Keep on doing this process we get that we can step up any stair. The property (II) can be illustrated in the same way. The property (III) means that we can step up stair 0, and in all cases when we cannot step up stair k, then there exists l with l<k that we also cannot step up stair l. From this follows that we can step up any stair. We can step up stair 1. If we could not do it, then we could not step up stair 0. If we step up stair 0 and stair 1, then we can step up stair 2,…, and so on.

Theorem 2. The properties (I), (II) and (III) are equivalent.

Proof. Firstly, we show that the property (II) follows from the property (I). Denote an arbitrary statement by P'(n). Suppose that (I) holds for P'(n) with $n \in \mathbb{N}$. Let P(n) be a statement depending on n with $n \in \mathbb{N}$, and for which (II)(a) and (II) hold. We will prove that P(n) is true for every n. Define the statement Q(n) with $n \in \mathbb{N}$ as follows:

$$Q(n) = P(0) \wedge ... \wedge P(n).$$

Thus, by Q(0)=P(0) , Q(0) is true. If $Q(k) = P(0) \wedge ... \wedge P(k)$ is true for some k with $k \in \mathbb{N}$,then P(0), P(1) ,…, P(k) are true. Therefore from (II)(b) we get that P(k+1) also is true. Then $Q(k+1) = P(0) \wedge ... \wedge P(k) \wedge P(k+1)$ is true. Hence, by (I) we get that Q(n) is true for every n. Then P(n) also is true for every n.

Now we show that the property (III) follows from the property (II). Let P'(n) be an arbitrary statement. Assume that (II) holds for P'(n) with $n \in \mathbb{N}$. Let P(n) be a statement depending on n with $n \in \mathbb{N}$ and for which (III)(a) and) (III)(b) hold. We will prove that P(n) is true for every n. It is enough to prove that (II)(b) holds. If (II)(b) was not hold, then this would mean that there exists t with $t \in \mathbb{N}$, for which P(0), P(1) ,…, P(t) are true, but P(t+1) is false. Then (III)(b) would not hold. This is a contradiction. Thus (II)(b) holds, and we get that P(n) is true for every n.

Finally, we prove that the property (I) follows from (III). Let P'(n) an arbitrary statement. Suppose that (III) holds for P'(n) with $n \in \mathbb{N}$ .Let P(n) be such a statement depending on n with $n \in \mathbb{N}$, and for which (I)(a) and (I)(b) hold. We must prove that P(n) is true for every n. Now it is also enough to show, that (III)(b) holds. If (III)(b) would not hold, then this would mean that there exists k with $k \in \mathbb{N}$ for which P(k) is false, but for every l with l<k the statement P(l) is true. Therefore there would exist such a k with $k \in \mathbb{N}$ for which P(k) is false, but P(k-1) is true. Then (I)(b) would not hold. This is a contradiction. Thus (III)(b) holds, and we get that P(n) is true for every n. The proof is complete.

The well-ordering property on the set of natural numbers is as follows:

Every nonempty set of natural numbers has a least element.

If we introduce the Peano axioms, then from Peano axiom 5 we can prove the following:

Theorem 3. The well-ordering property follows from Peano axiom 5.

Proof. Let A be with $\emptyset \neq A \subseteq \mathbb{N}$. If $0 \in A$, then 0 is the least element of A. If $0 \notin A$, then let

$$K = \{n : \ n \in \mathbb{N}, \ n \leq x \text{ for every } x \text{ with } x \in A\}$$

be. Since $0 \in K$, therefore $K \neq \emptyset$. If $x \in A$, then $x+1 \notin A$. We show that there exists element c for which $c \in K$ and $c+1 \notin K$. We use indirect proof. Suppose that, if $k \in K$ then $k+1 \in K$ for every k. From Peano axiom 5 (see: Remark) we get that $K = \mathbb{N}$ which is a contradiction. Thus there exists element c with $c \in K$ and $c+1 \notin K$ .Now we show that c is the least element of A. Really, $x \in A$ with $c \leq x$ for every x. If $c \notin A$ was, then $c \leq x$ would be from which $c+1 \leq x$ for every x. We get that $c+1 \in K$ which is a contradiction. Therefore $c \in A$, and c is the least element of A.

If we suppose that the well-ordering property is true, and we don't use the Peano axioms, then we get the following theorem:

Theorem 4. The validity of the property (I) follows from the well-ordering property.

Proof. Suppose that P(0) is true; and if P(k) is true, then P(k+1) is also true for all natural numbers k. To show that P(n) must be true for all natural number, assume that there is at least one natural number for which P(n) is false. Then the set A of natural numbers for which P(n) is false is nonempty. Therefore by the well-ordering property A has the least element which will be denoted by l. We know that l cannot be 0, since P(0) is true. Since l is a natural number and it is greater than 0, therefore l-1 also is a natural number. l-1 is not an element of A, so P(l-1) must be true. Since (I)(b) holds, it must be the case that P(l) is true. This is a contradiction of choice of l. Hence, P(n) is true for every natural number n.

Denote S(n) with $n \in \mathbb{N} \setminus \{0\}$ the following statement which is called generated Dirichlet box-principle:

If at least nm+1 objects $(m \geq 1)$ are distributed among n boxes, then one of the boxes contains at least $m+1$ objects.

Theorem 5. The statement $S(n)$ is true for every n.

Proof. 1. The statement $S(1)$ is obviously true.

2. Suppose that S(l) with $l > 1$ is false. Using the infinite descent method (property (III)) it is enough to prove that S(l-1) is also false. If S(l-1) was true, and we would give an another box with m objects to l-1 boxes, then altogether we would have l boxes and S(l) would be true. This is a contradiction.

Recursive definition

Using the Peano axiom 5 we get from the following theorem that is the simplest form of recursive definition and it is correct.

Theorem 6. Let $S(n)$, $(n \in \mathbb{N})$ be a notion (sequence) for which the following is true:

(a) the definition of $S(0)$ does not depend on any $S(n)$, $n \in \mathbb{N} \setminus \{0\}$, and $S(0)$ exists and it is uniquely determined,

(b) for arbitrary natural number k the definition of $S(k+1)$ depends on $S(k)$ at most, and knowing $S(k)$ we can see that $S(k+1)$ exists and it is uniquely determined, then the notion (sequence) $S(n)$ with $(n \in \mathbb{N})$ is well-defined, namely for arbitrary n. There exists only one $S(n)$, $(n \in \mathbb{N})$ which corresponds to the recursive definition.

Proof. Let T be the following property:

The natural number n has the property T if and only if $S(n)$ exists and it is uniquely determined. From (a) we get that 0 has the property T. According to (b) we have the following:

If the natural number k has the property T, then the natural number k+1 also have the property T , and it is true for every k. From Peano axiom 5 it follows that every natural number n has the property T. Therefore $S(n)$ with $(n \in \mathbb{N})$ is well-defined.

References

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