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IS ALGEBRA REALLY DIFFICULT FOR ALL STUDENTS?

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Abstract. Research studies have shown that students encounter difficulties in transitioning from arithmetic to algebra. Errors made by high school students were analyzed for patterns and their causes. The origins of errors were: intuitive assumptions, failure to understand the syntax of algebra, analogies with other familiar symbol systems such as the English alphabet and interference from arithmetic. There were other psychological factors such as carelessness, anxiety, overconfidence, and lack of motivation. Three major error types are discussed with their causes using a cognitive psychological approach. Solution methods of another group of students show that they were eager to use algebraic methods over arithmetic procedures even though arithmetic procedures are more straightforward. The paper argues that creative methods used by some students should be used to reinforce the learning of other students.

Keywords: Algebra, error analysis, misconceptions in algebra

Introduction

In recent years, many research projects on mathematics education have focused on learning difficulties of students related to algebra. Research suggest that solutions to the problem of student inability to be successful in algebra are many and frequently interconnected (Norton & Irvin, 2007). After studying several research findings, Norton & Irvin (2007) suggested some solutions which include: making explicit algebraic thinking inherent in arithmetic in children's earlier learning (Lins & Kaput, 2004; Warren & Cooper, 2006), explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997, 1999; Stacey & Chick, 2004), especially in transformational activities (Kieran & Yerushalmy, 2004; Stacey & Chick, 2004), using multiple representations including the use of technology (Kieran & Yerushalmy, 2004; Van de Walle, 2006), and recognizing the importance of embedding algebra into contextual themes (National Council of Teachers of Mathematics, 1998; Stacey & Chick, 2004). Other research have shown that student errors in algebra can be ascribed to fundamental differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from certain arithmetical conventions and need to learn to deal with algebraic symbolism.

The psychological approach to study cognition

Psychology plays an important role in mathematics education especially in cognitive analysis of children's mathematical thinking. The disciplines of psychology and cognitive science have a seminal influence on how mathematics is learned and taught. The contemporary psychology starts with the concept of an abstract mind. This paradigm implicitly makes the claim that the mind is universal. According to this view, local conditions like settings and cultures are external to the mind and detached from it. Contemporary psychology also believes that mind can be studied explicitly with the use of appropriate psychometric methods without any priori value judgments in seeking for what's going on in the mind. One of the pioneers of this view was Jean Piaget. He regarded learning as being subordinate to cognitive development and viewed the development of knowledge as a spontaneous process (Piaget, 1970). Piaget's explanation of the idea that children learn from reflecting on their actions is widely accepted even today. The constructivist perspective, derived in part from the work of Piaget asserts that conceptual knowledge cannot be transferred from one person to another. Rather, it must be constructed by each child based on his/her own experiences. The concept accounts for the

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individual idiosyncratic construction of meaning for systematic errors, misconceptions, and alternative conceptions.

The cultural historical approach to study cognition

In this view, learning is considered to be the process of constructing understanding out of the activities embedded in the social and physical world. Knowledge can be viewed as a product of the activity and situations in which they are produced and used (Lane, 1993). In this approach, learning occurs as a process within ongoing activities or a process involving culturally mediated and historically developed practical activities. Therefore, knowledge acquired through learning is not purely situated in an individual's brain, but one should look at cognition as a historically situated, culturally mediated, and socially organized activity.

Cole (2005) argued that cross-cultural research conducted before psychology came into being do not adequately explain human cognitive development. He further said that the strategies used in standardized cross-cultural research or multidisciplinary research on cognitive development also have the same shortfall of inadequacy to explain cognitive development of humans. Therefore, a "cultural-historical" approach should be considered as an effective way of studying cognition that mixes with contemporary ideas in anthropology and cognitive science.

Contemporary psychological perspective of the analysis of cognitive activities focuses on individuals' construction of knowledge based on their previous experiences. On the other hand, cultural historical analysis focuses on knowledge as shared. This view further assumes that children construct knowledge through cultural historical shared activities and artifacts. It employs the differences in contexts, teachers' and students' beliefs, and their motives and norms to examine human cognition.

In the next section, I will explain the results of students' algebraic reasoning using the psychological view of cognition as a basis. Here, I assume that individuals could explain their thinking based on what they have constructed individually in the classroom learning process. However, it is not possible to explain all the underlying causes of student behaviors using this single approach. Studying students' errors is a complex process. We will not be able to explain certain kinds of student behaviors using a single perspective. Based on another set of student answers, it is argued that it will be better to use multiple theoretical approaches to identify student thinking more meaningfully.

Methodology

The first part of this study was carried out with a group of 120 high school students in grades 9 and 10 in Sri Lanka. The three schools selected were from urban, semi-urban, and rural areas in order to get a group of mixed ability students. An algebra test with 30 items which has been pilot-tested was administered to all students. The students were from a mixed-ability group in order to ensure the identification of a sufficient number of interviewees, since higher-ability groups would presumably contain fewer students making errors. A sample of 32 students was later selected for interviews after two weeks of the test. All care was taken to ensure that the sample included the largest possible variety of errors. Each student was interviewed individually and the results were tape-recorded. Each interview lasted within 15 to 30 minutes. The interview schedule was adopted from Newman (1977), Casey (1978), and Clements (1980).

The interview transcripts in conjunction with students' written work were analyzed seeking for patterns and regularities. In this paper, three prominent error types and their causes are discussed. The discussion in the second part of the paper is based on some algebraic methods used by college students. These students are from a hospitality mathematics program and many of them had recently completed their high school education. This discussion is based on the symbolic methods used by two students in solving word problems.

Discussion of the results

Initially, the errors made by students were categorized according to the stages of the problem solving process in the Newman model. The greatest number of errors occurred during the processing stage (57.8%) followed by comprehension error (21.9%), encoding error (15.6%), and verification error (4.7%). These figures indicate that nearly 80% of the errors had occurred during the comprehension and processing stages. The balance 20% had accounted during encoding and verification. The areas where major error types found were: the transformation of word problems into algebraic language (49.4%), parenthesis omitted (38.7%), and wrong operations in solving equations (29.2%).

Error category and sample problem	Errors and their possible causes
1.**Transformation of word problems into algebraic language (49.4%)** **Question:** Amanda is 4 years older than Diana. The sum of their ages is 28 years. What is Amanda's age?	• Attempted to translate the problem word by word or phrase by phrase into algebraic language causing "reversal errors". • Unable to understand the numerical relationship between the two quantities in algebraic terms due to poor understanding of the arithmetic-algebraic connection in the problem. • Unfamiliarity of the existence of an algebraic entities such as $4+x$ and conjoining them as $4x$.
2.**Parenthesis omitted (38.7%)** **Question:** The length and the width of a rectangle is given by $b+2$ and 5 respectively (a figure was provided). What is the area of the rectangle?	• Lack of understanding of the precedence of operations with algebraic terms. • Do not see the need of brackets and attempt to simplify an algebraic expression from left to right operating on immediate terms possibly because of the interference from previously learned wrong concepts with numbers. • Difficult to perceive numbers, signs, and letters together as expressions. Try to separate numbers from letters and carry out operations due to difficulties of understanding the syntax of algebra.
3. **Wrong operations in solving equations (29.2%)** **Question:** If $x=y+z$ and $x+y+z=30$, then find the value of x.	• Assign arbitrary convenient values for variables. • Lack of ability to generate and perceive a global overview of different parts of an equation including the equal sign due to poor understanding of the structural aspects of algebra. • Lack of understanding of the difference between letters as specific unknowns and letters as generalized numbers. • Other non-algebraic errors such as dropping a sign, incomplete working, transcription errors, and other computational errors due to psychological factors such as low attention to the task, carelessness, or lack of motivation.

Research suggests that in general, symbolic manipulation is difficult for many students. However, this was not the case for some college students in this study. They preferred to use algebraic methods rather than arithmetical methods to solve word problems. They admitted that algebraic methods are easier for them as some arithmetical methods need them to think backwards. The following example is a sample of work of one student.

Problem	Strategy used
1. A 900-room hotel had some of the rooms closed for renovations last week. 32% of the ***available*** rooms were sold on a particular night. a) How many rooms were available each night last week, if 96 rooms were sold?	a) x – available rooms – unknown 32% x = 96 32/100 x = 96 x = 300

b) What percentage of its usual total number of rooms was available last week?	b) 300/900 % = 33.33%
2. A hotel offers a 15% discount on all double-occupancy rooms. Determine the original price of a room, if a discounted room sells for $81.60.	Discounted price – 81.60 Original price – x Discount amount – 15% x = 81.60 + 15% x x = 81.60 + 15/100 x x – 3/20 x = 81.60 20 x – 3 x = 1632 17 x = 1632 x = 96 Proof: 15% of 96 = 96*15/100 = 14.4 96 – 14.4 = 81.6
3. If 1 lb. = 454 g, Calculate the number of grams in one ounce.	$\frac{16oz}{1} = \frac{454g}{x}$ $x = 28.375g$
4. A restaurant bought domestic Riesling wine at $ 3.20 per bottle. Each bottle contains 8 pints. The wine is served in 5 oz. glasses. The restaurant is to serve a party of 60 guests. Each guest will receive one glass of wine. Calculate the restaurant's cost for one fluid oz. of wine. (1 pint = 20 fl. oz.)	$\frac{160}{1} = \frac{3.20}{x}$
5. A set of kitchen tools retails for $ 450. The discounted price is $ 360. Calculate the percentage of the discount.	$100 - x\% = \frac{360}{450} \times 100$ $-x = 0.8 \times 100 - 100$ $x = 20\%$

The student, who answered question 1, has selected to use algebraic methods over arithmetical methods. Arithmetically, the question in part (a) could be asked as "If 32 % of some number is 96, then what is the number?" Simply, the calculation would be $96 \times \frac{100}{32}$. Using the same argument, the answer to question 2 would be $81.60 \times \frac{100}{85}$. The student, in questions 3 and 4, has correctly identified the relationships among the variables and has derived a simple ratio to solve the problem using an algebraic method rather than simply carrying out the conversions by dividing numbers.

It is true that these solutions are not very innovative. However, the methods are creative. These students have a better understanding of algebraic relationships and the concept of variables than the students who use arithmetical methods. When I interviewed the first student for her use of methods, she said, "When reflecting back on my education, my interest in mathematics started when I began to learn algebra in my first year of high school in Italy. Even at that time, I did not like arithmetic very much. I was thinking of a way to get rid of these cumbersome arithmetical methods. It is hard to think in terms of arithmetical principles. They made no sense to me. It is easier to think in terms of letters. I believe that I have strong logical skills. After my high school, I studied law for some time before coming to Canada. Algebra made mathematics simpler for me because I could see the patterns and the structures of algebra. Actually speaking, the generalization in algebra is much clearer to me than the detail of arithmetic."

The reasons for students to use algebraic methods over arithmetic methods have many aspects. Sometimes teachers use these methods in their classrooms and subsequently this will have an influence on their students. A study carried out in Belgium reported that those who taught high school preferred to use algebraic methods for solving arithmetic and algebraic problems (Van Dooren et al., 2002). They used algebraic procedures even when arithmetic methods were more straightforward. Prior research also established that high school teachers with high subject-matter knowledge favor a symbol precedence view of algebraic development. However, the results may also have an influence from curriculum demands, textbook structure, and possibly many other factors.

Conclusion and recommendations

Algebra is hard to teach and hard to learn. As MacGregor & Stacey (1997) pointed out, difficulties in learning to use algebraic notation have several origins. The cognitive obstacles that learners encounter in engaging mathematical tasks include, but are not limited to, student difficulties with syntactic/semantic integration, difficulties in recognition of applicable solution templates, vague assumptions regarding variable choices, and imprecise algebraic representations of word problems. Word problems contain inherent cognitive obstacles for students as these problems must be interpreted from a syntactic form into a semantic conceptualization, and then again into a symbolic generalization of the situation. The students in this study belonged to two different categories. Some of them use algebraic methods very efficiently while others struggle to cope with them. The question is whether cognitive factors, culture, previous teaching, or any other factors have an influence on student methods. Whether the protocol analysis alone will adequately explain the full reasons behind student errors and their selection of methods is another issue.

Adopting the Verbal Analysis method for analyzing protocols of classroom lessons to uncover mental representations could sometimes be problematic as classroom protocols include participation of multiple agents. Hence, it is difficult, if not impossible, to isolate an individual to study his/her mental representations or learning processes. Therefore, using more than one theoretical framework for analyzing students' solutions is important. It could be that students erred merely because they were careless or have used incorrect multiple thinking strategies. Or the reason may be that they were taught differently. Therefore, it is important as well as theoretically sound to use a variety of approaches when analyzing data regarding students' mathematical reasoning. A cognitive analysis will point to students' cognitive difficulties in addressing different problems while a cultural historical analysis will point to differences in the teachers' and students' motives, their beliefs, and norms regarding mathematics that shapes their methodology. Therefore, a mixture of cognitive and cultural historical approaches will be a better choice. One theory may not provide all the answers regarding all sources of errors.

The findings underline the importance for teachers to have a deeper knowledge of mathematical content as well as insights into student thinking. Teachers must identify the fundamental ideas that need to be taught, and must understand the difficulties and misunderstandings that are likely to occur. Problems that can be solved using algebra have different structures and processes. Using the same method to teach every student may not always work. Some diagnostic teaching is sometimes necessary. When students are given opportunities to verbalize their mental processes, it would mostly facilitate their transfer from arithmetic to algebra. However, this transition is neither automatic nor smooth. Sometimes, the structural differences of the two systems will interfere for a smooth transition process.

Classroom teachers should focus their attention on students who lack the capabilities of transition from arithmetic to algebra and also on students who use algebraic methods to solve problems that could have been solved using simple arithmetic. Probably, the latter group of students may have developed fluency and symbol sense than the former group that led them to operate with algebra as a language and as a set of problem solving methods. These students undoubtedly have relatively strong symbolic and logical skills. In classrooms, they need to be given a stronger algebra diet and they should be encouraged to make "think aloud" about their methods whenever possible in order to get insights about their methods. If we, as researchers, could situate those methods in a broader framework of error analysis, then we will be able to exploit these methods for the benefit of those who lack these skills.

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