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# LEARNING MATHEMATICS, DOING MATHEMATICS: DEDUCTIVE THINKING AND CONSTRUCTION TASKS WITH *THE GEOMETER'S SKETCHPAD*

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**Abstract.** A deductive thinking can be considered as the concatenation of ideas, each one determined by the previous one. In mathematics, deduction is the way in which we validate a conjecture, using general facts to justify less general or particular facts. All individuals possess this way of constructing thoughts and jumping into conclusions, but we need to practice it in order to develop it deeper and make more educated decisions.

In construction tasks with The Geometer's Sketchpad (GSP) it is possible to explore the construction itself, make conjectures and try to validate them, putting ourselves in a theoretical context and making use of deduction as the means of getting the right answers.

Learning Mathematics, Doing Mathematics, is a teaching model in which I propose, among other things, the fostering of a deductive thinking mainly through construction tasks in Euclidian Geometry. In this paper I present the main features of the model and some GSP activities that are helpful in the fostering of a deductive thinking.

**Keywords:** learning Mathematics; Geometer's Sketchpad; deductive thinking

## 1. Introduction

*Learning Mathematics, Doing Mathematics* (Flores 2007a) is a teaching and learning model aimed to foster in students the following:

• A mathematical thinking that allows to distinguish patterns and generalize; justify results with mathematical arguments; and use several representations of a same mathematical object.

• Problem solving skills that allow posing and solving problems inside or outside a mathematical context.

• Technology competence that allows the use the technology at hand in order to facilitate problem solving and the acquisition of knowledge.

• Positive attitudes toward mathematical tasks. These attitudes allow posing and solving problems as students' own responsibility and will result in his benefit and the benefit of others.

• Human values that allow a better social and environmental coexistence.

These five issues are achieved through the implementation of a suitable Teaching and Learning Environment (TLE) where the student feels responsible for the acquisition of his knowledge. In this TLE the whole class fosters basic mathematical knowledge and skills, and promote positive attitudes

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and values toward each member of the class. The TLE is divided in two main aspects: Competencies and Personal Qualities

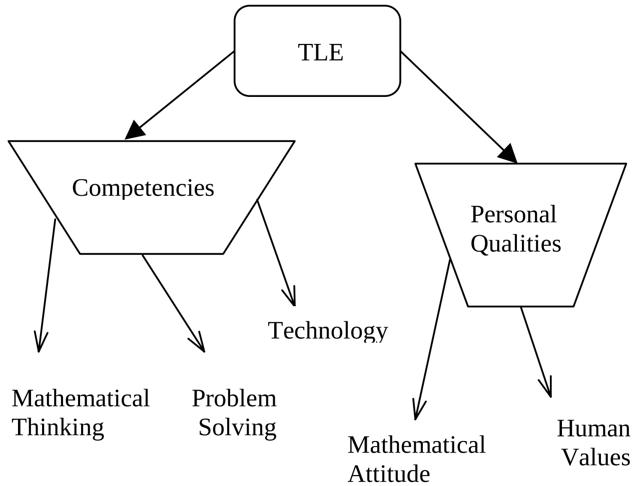


**Figure 1.** TLE for learning Mathematics, doing Mathematics

The two aspects of the TLE complement each other, that is, if a student has human values that allow him to interact harmonically with his classmates and teacher in a environment of cooperation, tolerance and respect, is very likely that he develops a mathematical thinking that, in turn, allows him to solve problems; and if he thinks mathematically, solves problems, and is able to use the technology at hand, he will have more elements to cooperate and help his classmates.

The competence aspect of the model is developed through three types of learning activities: exploration, problem solving, and non-routine problems.

Generally, **exploration** activities are given in Euclidean Geometry tasks. Here the student should make some conjectures in order to answer a question or to construct some geometrical configuration. The student should validate these conjectures. The following is an example of this kind of activities.

*Construct a square and its diagonals. How are the four inner triangles compared to each other? Explain.*

When a student tries to explain the answer puts into play his argumentative schemes. We have found that most of students in the first year of High School use empirical schemes (Flores, 2007b, p. 48), that is, measure the triangles and explain according to the outcome; some of them use a scheme that is in the middle between an empirical scheme and an analytical one, for instance, they say that the sides of the triangles that coincide with the sides of the square are congruent because the four sides of the square are congruent to each other, then they take a compass and check that the other sides are radii of the same circle; and very few of them use a deductive reasoning in an analytical scheme.

**Problem solving** activities are, mostly, modeling problems. Here the student should make decisions about the mathematical model (generally a mathematical function) that better suits the situation. For instance:

*The depth of water in a shore varies with time due to tides. In certain shore we have a high tide every 12 hours and the depth, 15 meter from the beach, varies between 1.5 and 2.00 meters. Find a function that models water depth depending on time.*

Here the student must make several decisions, as what kind of function to take and where to put the origin of the movement.

**Non-routine problems** are situations in which the student must use his knowledge and imagination. For instance:

*17 houses are located along 1 kilometer of highway in irregular intervals. Where should a bus stop be located in order that the sum of the distances from each house to the stop is minimal? Explain*

Here the student should use his strategies of problem solving and his ability of pattern recognition and generalization.

With the teaching activities is possible that the student learn mathematics, doing mathematics. But, in order to achieve this, we should work in an environment of harmonious coexistence and cooperation. This has to do with the Personal Qualities aspect of the teaching model. In this context, we consider the class as a knowledge community; its members have common learning goals and the way to obtain them is working together with tolerance, cooperation and respect.

In a Learning Mathematics, Doing Mathematics TLE, the student learn Mathematics in the same way as a learner in a vocational workshop; that is, involving directly in the tasks according to his experience and knowledge. Thus, with the help of the teacher and his classmates, the student is becoming expert in his vocation: Mathematics. In our teaching model, student works in teams; we encourage an open communication between members of the teams, between teams, and with the teacher. Teacher interventions generally have the goal of feedback the process. Teacher monitors and encourage student's work giving the suggestions and advise to achieve the objective. Table 1. (Flores, Gómez, 2009)

**Table 1.** Relationships between TLE elements.

| **Qualities** / **Competencies** | **Mathematical Attitude** | **Human Values** |
|---|---|---|
| **Mathematical Thinking** | Pattern recognition and generalization<br>Construction of deductive chains<br>Understanding the difference between an empirical demonstration and a mathematical proof.<br>Awareness of achievement and limitations. | Opportunity to explain and justify one's position regarding a specific answer. .<br>Development of confidence in our own work. |
| **Problem Solving** | Use of a logical or mathematical structure when reasoning.<br>Validation of results.<br>Use of different representations. | Organized participation in teamwork.<br>Effective communication of outcomes.<br>Respect of ideas different of our own.<br>Confidence when facing unknown situations. |
| **Technology** | Visualization of mathematical situations.<br>Pattern recognition and conjecturing through exploration. | Confidence in managing technology.<br>Integration to different ICT (forum, chat, virtual classrooms, etc.) |

## 2. Deductive Thinking and Construction Tasks

### 2.1. Deductive Thinking and argumentation schemes

Somewhere else (Flores 2006, 2007b, 2007c), I found that High School teachers use different argumentation schemes when facing construction tasks with the Geometer's Sketchpad, namely (Flores 2009):

• *Authoritarian.* Here arguments are based on statements made by some authority –a teacher, a textbook, a principal, etc.

• *Symbolic.* The individual uses mathematical language and symbols in a superfluous or naive way, without really getting to the conclusions meant. The individual often mentions unclear or redundant concepts as *equilateral rectangle* or *regular trapezoid.*

• *Factual.* Here we have an account of the actions taken or a repetition of evident facts as an explanation or justification of a result. The individual, often, exposes a set of steps in an algorithmic way.

• *Empirical.* Here the argument is based on physical facts or drawings. In this case the fact or the drawing is an argument by itself and not a visual help of the argument.

• *Analytical*. Here the argument is a deductive chain in which a statement is conclusion of the previous one. This deductive chain not necessarily ends in a valid conclusion.

With some variations these same schemes are found in High School students. Let us see an example of the way in which High School students proceed in a *Sketchpad* exploration task, making conjectures and going to extreme cases in order to find an explanation (Dalcín, 2004, pp. 64-65).

*Construct a triangle ABC (Figure 2) and the external angles CBP, ACQ, and BAR. What can you say about the sum of these angles? Explain*

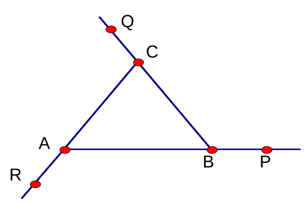


**Figure 2.**

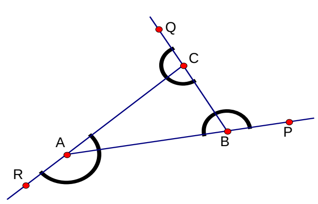


**Figure 3.**

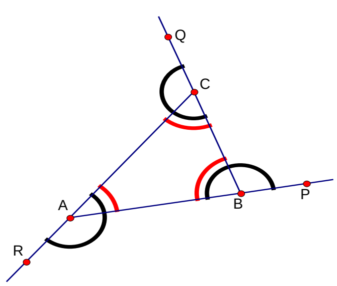


**Figure 4.**

***Javier–Matías****:* They construct a figure similar to Figure 3, measure, sum and get 360º, and verify that this 360º remain constant if they drag the vertices. They went in the next dialog when searching for an explanation

*J: In each vertex there are two joined: QCA y ACB for instance.*

Matías drags the vertices in such a way that he have an equilateral triangle*:*

*M: And if we make the triangle equilateral the outside angles are going to measure 120º because the inside ones measure 60º.*

*Thus, 120º by 3 gives us the 360º.*

*J: Yes, but if we drag the triangle does not work anymore. We need to do it for any triangle.*

*M: No matter if we distort it, we still have two joined angles –what you said at the beginning– and they form straight angles. So, if we want to find the sum, we make 180º x 3.*

Javier, pointing at the external angles on the screen (Figure 4):

*J: But what we want is to add up these ones.*

*M: And we subtract to each half turn [180º] the inside angle: 180º - CAB, for instance.*

*J And what we have? The number that we already have on the screen, the measure of angle RAB. Ah, wait! We can do what you are saying to the other two vertices.*

*M: We would have a turn and a half less...*

*J: ...minus the three ones from inside [the triangle].*

Simultaneously, Matías answering the question asked while he was speaking: minus which quantity?

*M: If it has to be 360º, it is a turn and a half minus half a turn'*

*J: The three inside angles.*

*M: Half turn.*

*J-M* (at the same time)*: The inside angles sum up half turn, 180º. That's it! We are geniuses!*

They write down on the worksheet, drawing a triangle and marking the three inner angles

$180^{o} \times 3 - (CAB + ABC + BCA) = 540^{o} - 180^{o} = 360^{o}$.

This couple used a deductive reasoning in order to explain their findings, that is, they used an analytical scheme. In Dalcin (2004) it is possible to find more examples of this kind of argumentation schemes.

### 2.2. Construction tasks with The Geometer's Sketchpad

With the aid of *The Geometer's Sketchpad* it is possible to design exploration activities in which student is able to make conjectures and try to explain their validity in a theoretical context. The use of this kind of software has the following features:

• Help in the transition from an empirical scheme to an analytical one (Hoyles and Jones, 1998; Gravina, 2000).

• Improve the understanding of the nature and aims of mathematical proof (Hoyles and Jones, 1998, Southerland, et. Al., 2004).

• Improve generalization skills in students (Pressmeg, 1999).

• Situate students in a theoretical context (Mariotti, 2000; Sutherland, Olivero and Weeden, 2004).

• Useful in eliminating conjectures that seem reasonable (Giamati, 1995)

• Contribute to the joint construction of knowledge (Mariotti, 2000).

• Work as a link between the phenomenological world and the theoretical one (Mogetta, 2001).

So, the use of Dynamic Geometry software in construction activities is helpful in making the students to use analytical argumentation schemes. This is so because, in order to explain the construction and why it works, they should explore the construction mostly using the drag test (as illustrated in the last subsection) looking for the certainty of their conjectures and for relationships among the different elements of the construction; and then searching for reasons that support their findings.

In the following I will illustrate further how construction tasks with *Sketchpad* allow the development and use of a deductive thinking on behalf of students. The activities reported are part of a 10th grade course, Euclidean Geometry, at the Colegio de Ciencias y Humanidades in Mexico City. Following the Learning Mathematics, Doing Mathematics Model, students work in pairs, but communication is not forbidden between pairs.

**Activity 1**

Construct a square using three different ways of doing it. In each case, explain why your construction is a square.

Generally, the first construction obeys to the definition of a square as a quadrilateral with congruent sides and angles. They start with a segment and rotate it and its end points twice by 90º, and then join two end points in order to complete the square. Students generally say that this is a figure with four sides of equal length and same interior angles.

But with subsequent constructions students have to appeal to some other properties of squares and explain their construction taking into account some other properties of squares, using their theoretical knowledge about them and using it in a deductive way (mainly through analytical schemes). Let us see, for instance, the next construction.

As most of the constructions it begins with a segment *AB*, then they construct a perpendicular line to that segment passing through one end point. After this, they draw a circle centered at the intersection *A* of the segment and the line. Having this, they join the intersection of the line with the circle, point *C*, with the other end point of the segment, point *B* (Fig. 5).

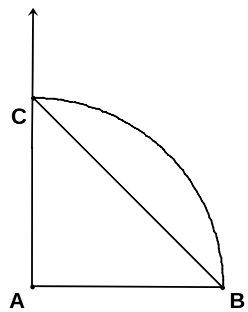


**Figure 5.** Activity 1

The next step is to reflect point A using segment *BC* as mirror, having, in this way, the four vertexes of the square.

One of the explanations of why it is indeed a square is in terms of the symmetry of the figure, more or less along these lines:

“We know that a diameter of a square is also a symmetry line, so if we reflect a right triangle with legs of the same length along its hypotenuse, we will have a square, that is, a figure with four equal sides and opposite angles of 90º.” This kind of reasoning corresponds to an analytical scheme.

Depending on the kind of construction and the square properties used, the explanations use different facts and relationships, in most of the cases using analytical schemes.

The next activity is another example of how can we foster analytical schemes or a deductive reasoning.

**Activity 2**

a) A quadrilateral whose diagonals intersect each other at its midpoint is a parallelogram. Is this statement true? Explain your answer. b) Construct the quadrilateral with *Sketchpad.* Write down step-by-step how the construction was made.

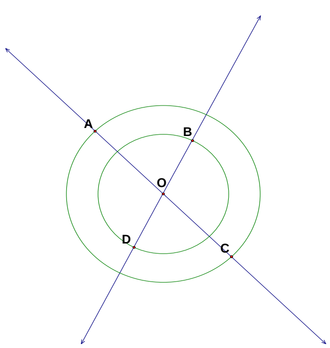


**Figure 6.**

Most students construct a parallelogram and then show that its diagonals intersect at their midpoints. This same reasoning was found in teachers (Flores 2007b).

Many students proceed more or less as follows. When the teacher question them about their reasoning, (*You have shown that a parallelogram has diagonals that intersect each other at their midpoints, but if you have two diagonals that intersect at their midpoints, these are diagonals of a parallelogram?*), they change the order and try to construct first the figure, beginning with two segments intersecting at their midpoints.

In order to construct the two segments they say that the center of a circle is the midpoint of its diameters, so if we have two concentric circles and take two diameters, *AC* and *BD* we should have the diagonals of the required quadrilateral. Figure 6.

This kind of reasoning is a deductive one, thus the scheme is analytical. A relevant point here is that the discussions in order to get the right figure puts students in a theoretical context.

These two activities are just examples of the way students could use their knowledge to get the explanations and how teacher help them to walk in the direction required, and how to use their reasoning in a more effective way.

In a typical *Learning Mathematics, Doing Mathematics* TLE, students work in teams and the teacher monitors the team proceeding with the activity and spread ideas and findings among the different teams.

## 3. Conclusion Remarks

After more than 10 years constructing and refining the model *Learning Mathematics, Doing Mathematics*, the empirical evidences gathered say that the model is effective in promoting a mathematical thinking and problem solving skills in technological and non-technological contexts, and is very useful in developing self esteem in students as well as some human values such as tolerance, respect and cooperation.

In particular, with exploration activities –like the ones presented in this paper-, with the Model it is possible that students develop a deductive thinking and make use of the theoretical facts involving problem solving.

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Professor A. Homero Flores is a full time mathematics teacher at Colegio de Ciencias y Humanidades-UNAM in Mexico City since 1998, and educational researcher since 2000. His main interest is on students and teacher argumentation schemes, modelling in learning and teaching, and teachers training at High School level.